Persistent Homology in Image Processing

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GbR'13 Vienna

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Herbert Edelsbrunner

IST Austria

Persistence	I	Hierarchy
Extended Persistence	II	Adaptive Topology
Stability	III	Measuring
Moments	IV	Scale Space

I.1 1D Function

function

$f : \mathbb{S}^1 \to \mathbb{R}$

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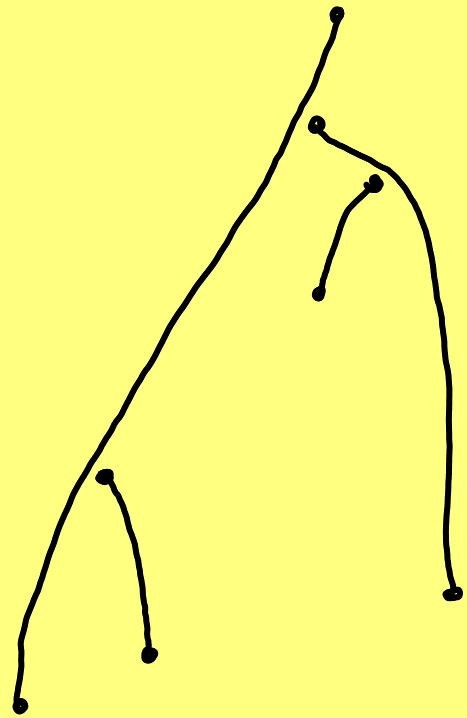

merge tree

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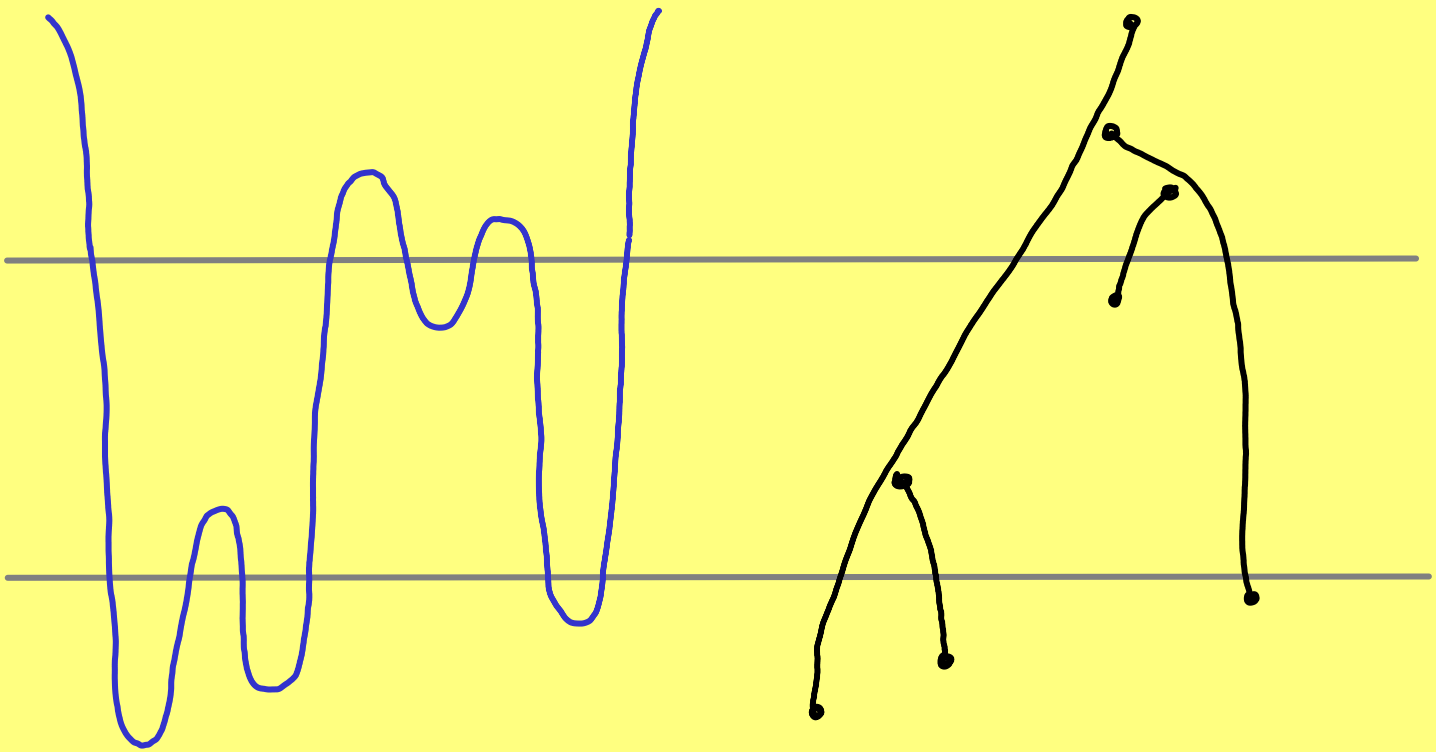

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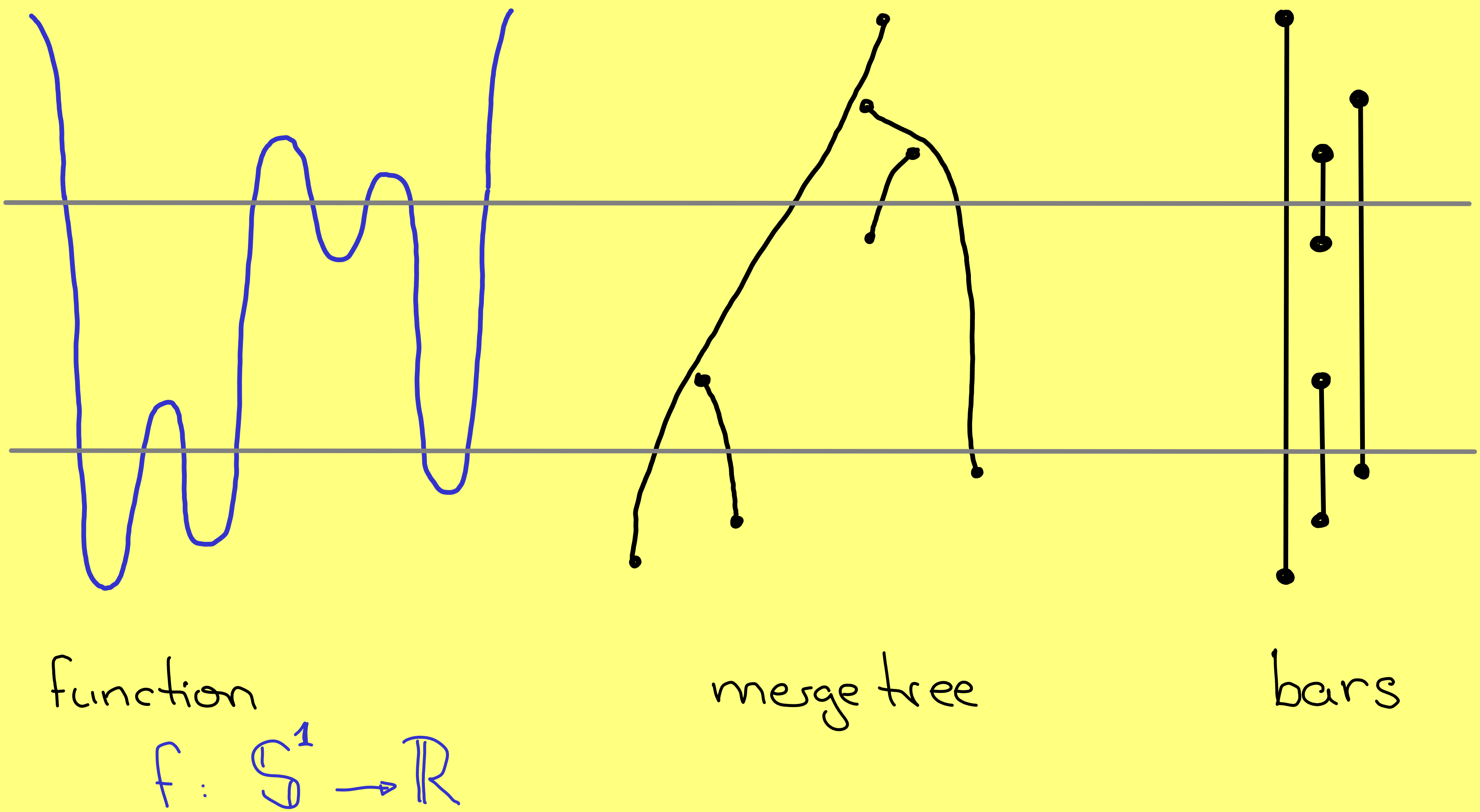

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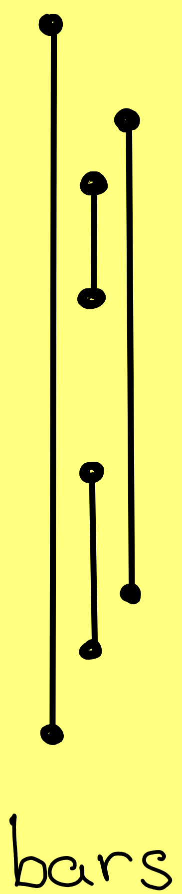

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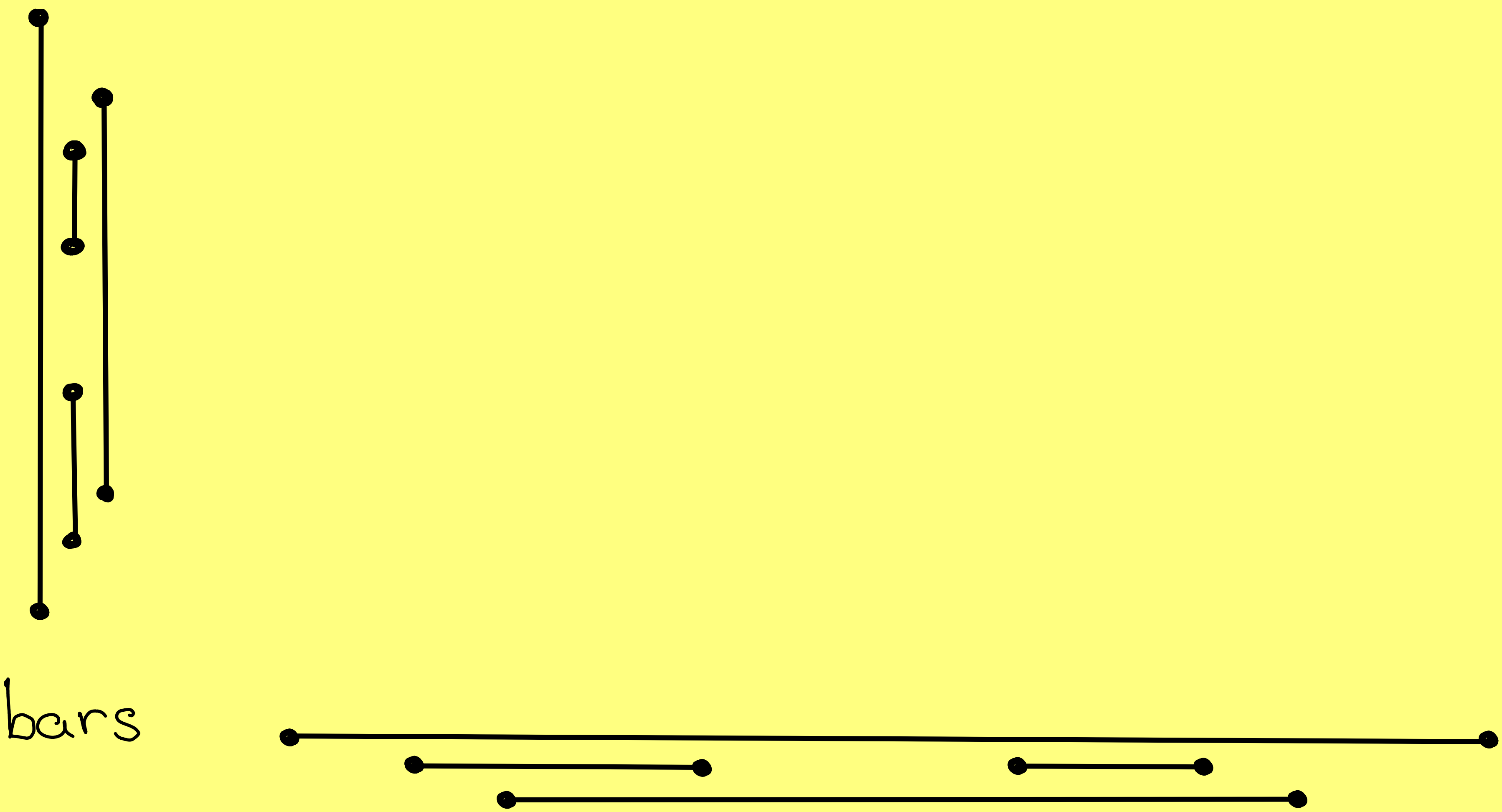

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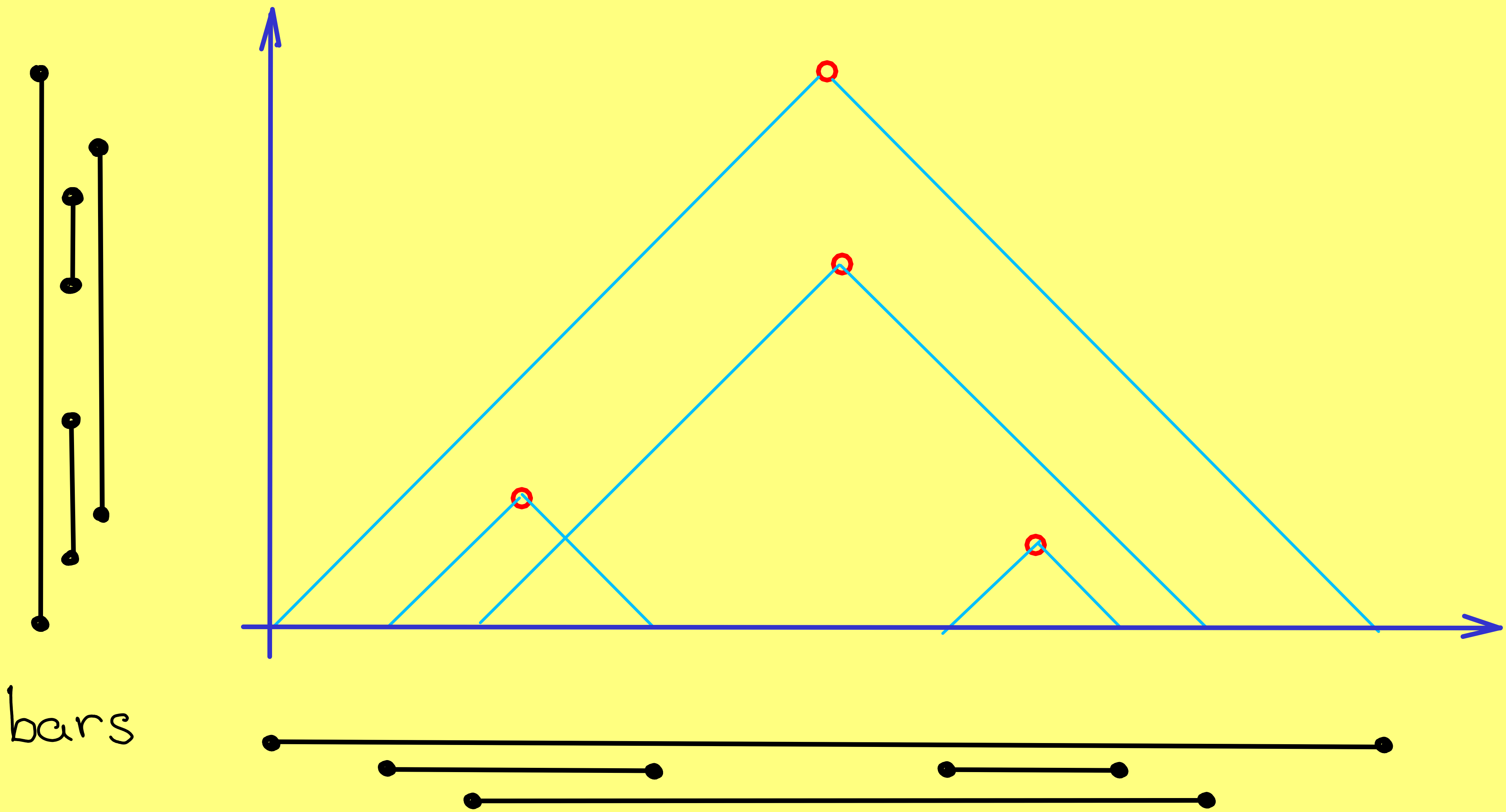

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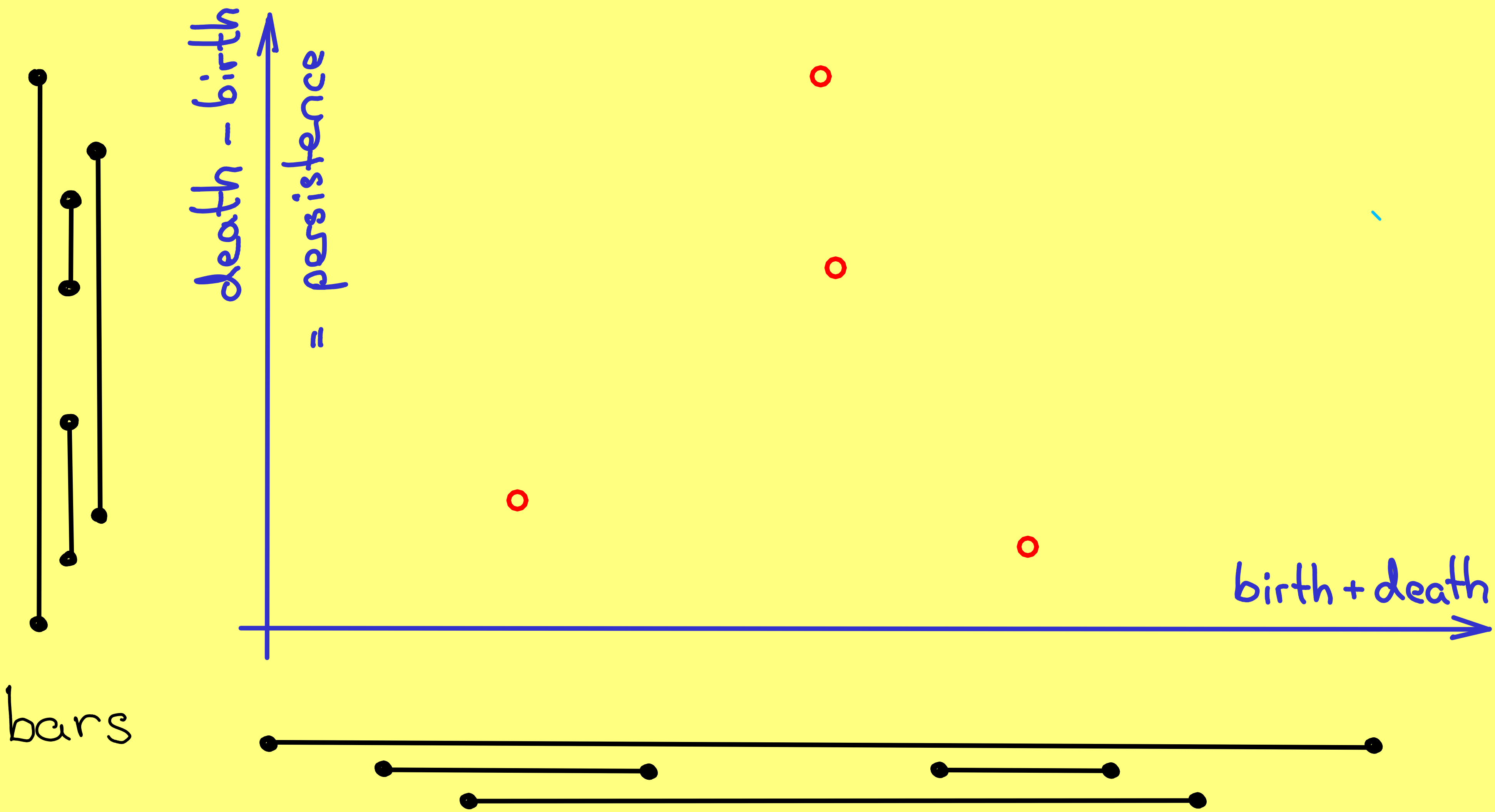

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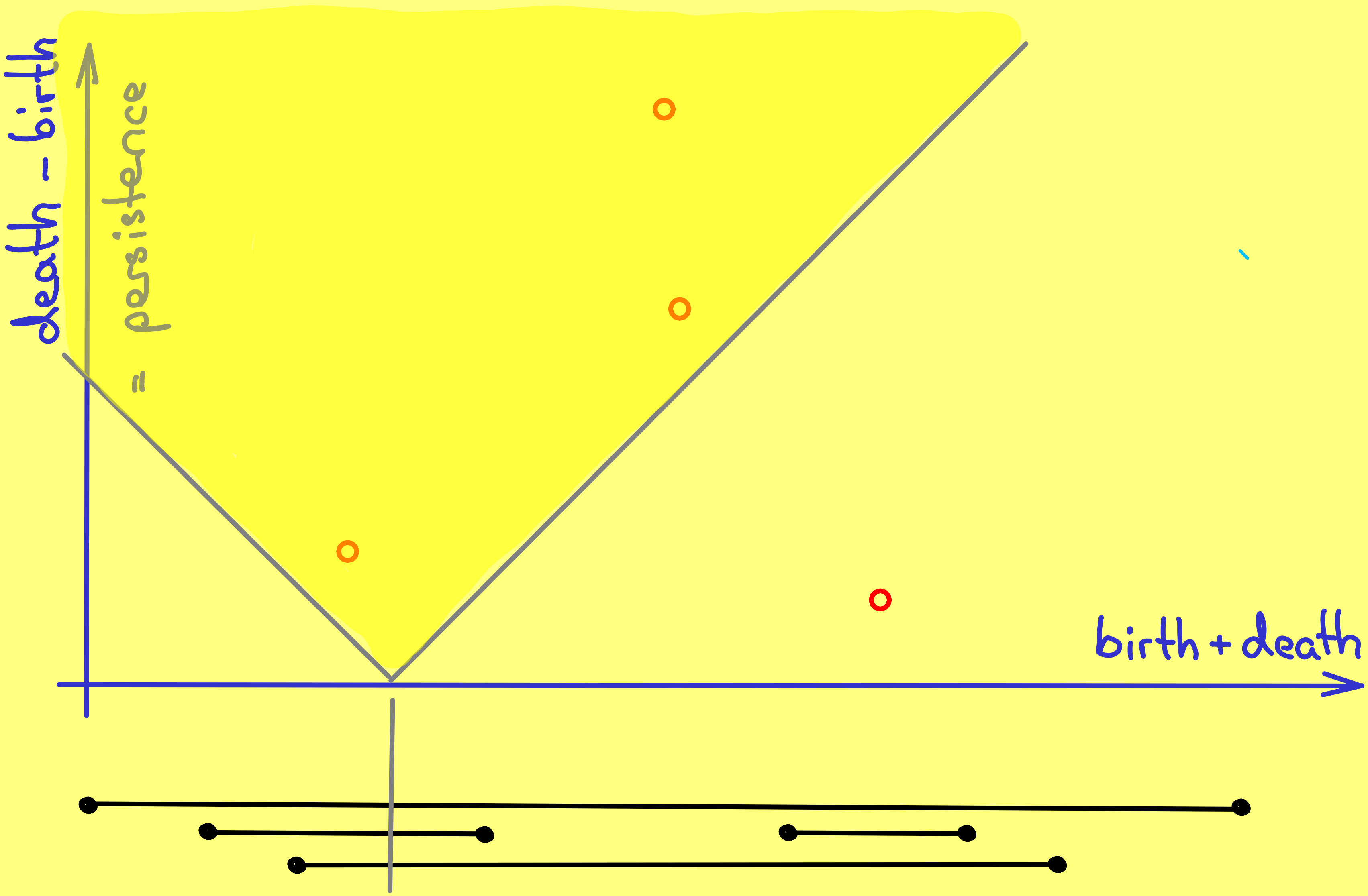

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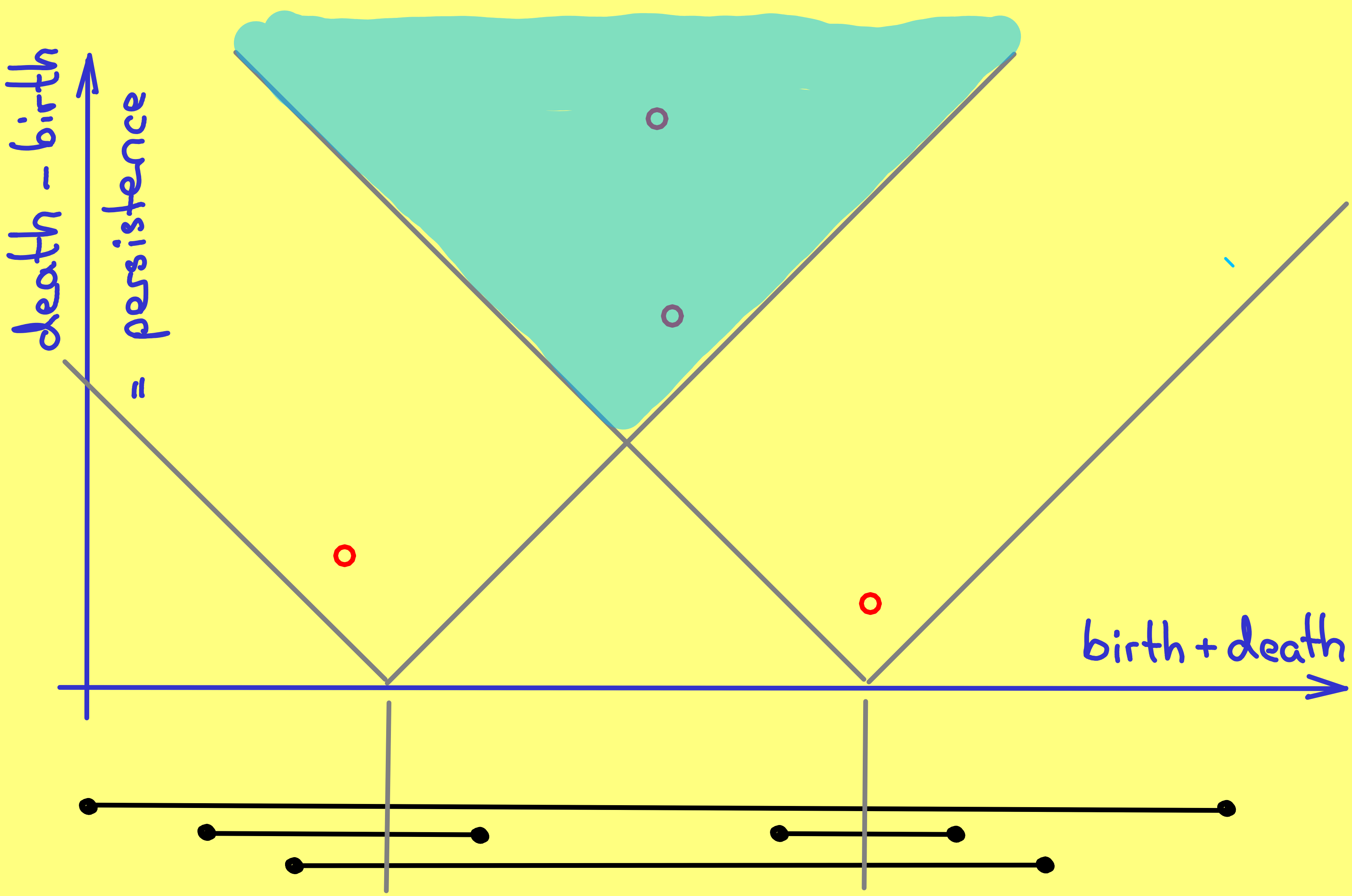

BRIEF HISTORY

FERRI, FROSINI 1991

size function = 0-dim. pers. hom.

E, LETSCHER, ZOMORODIAN 2002

introduction of pers. hom. as we know it.

I.2 2D Function

$$f : \mathbb{R}^2 \to \mathbb{R}$$

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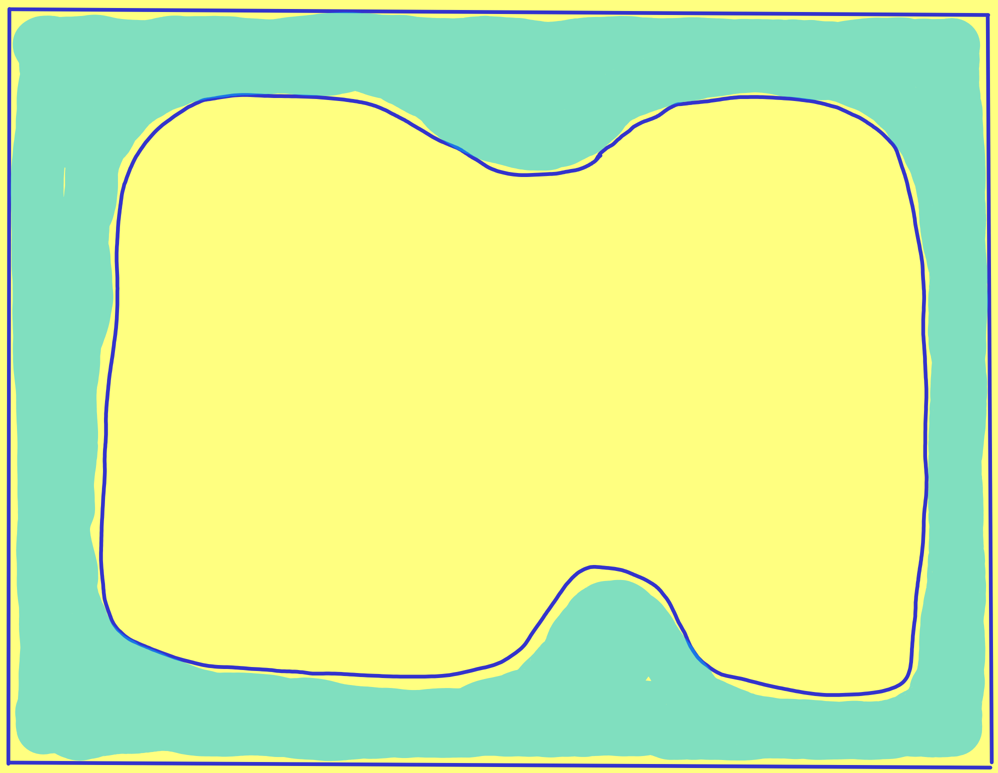
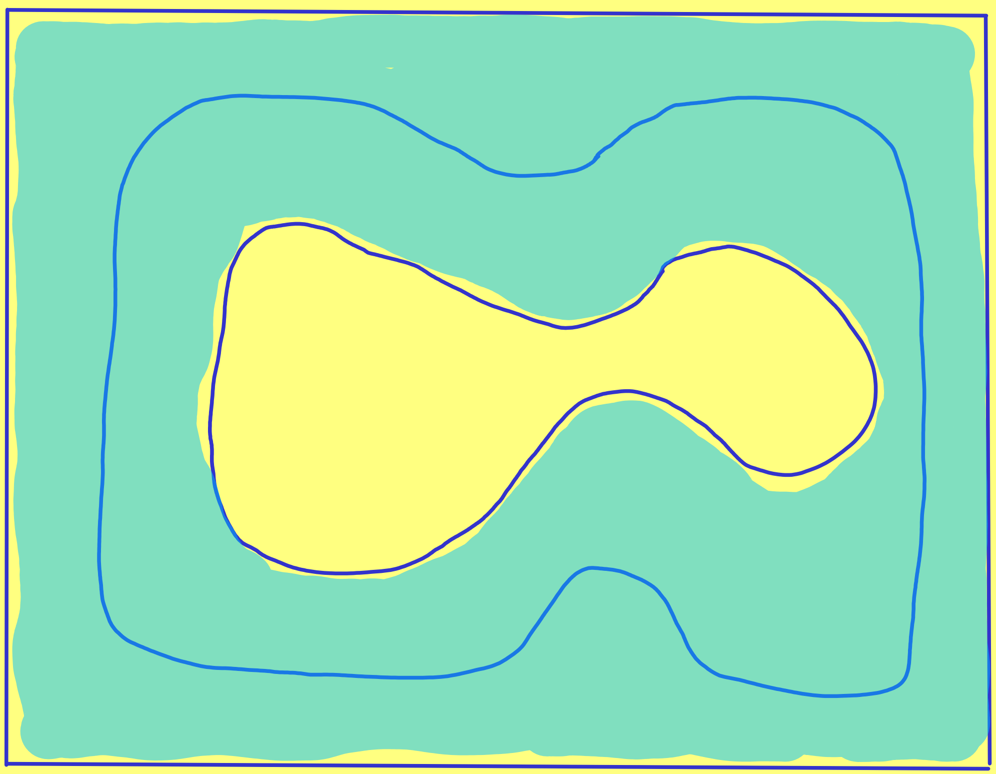
sublevel sets $f^{-1}(r_1)$



I.2 2D Function

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

sublevel sets $f^{-1}(r_1) \subseteq f^{-1}(r_2)$



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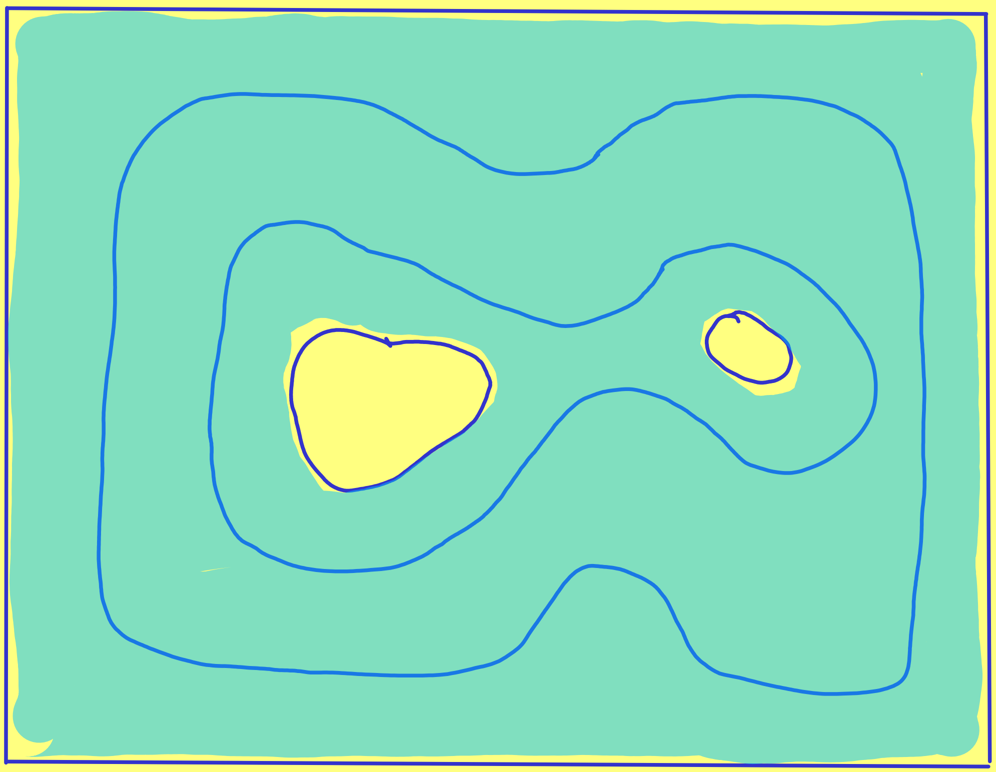

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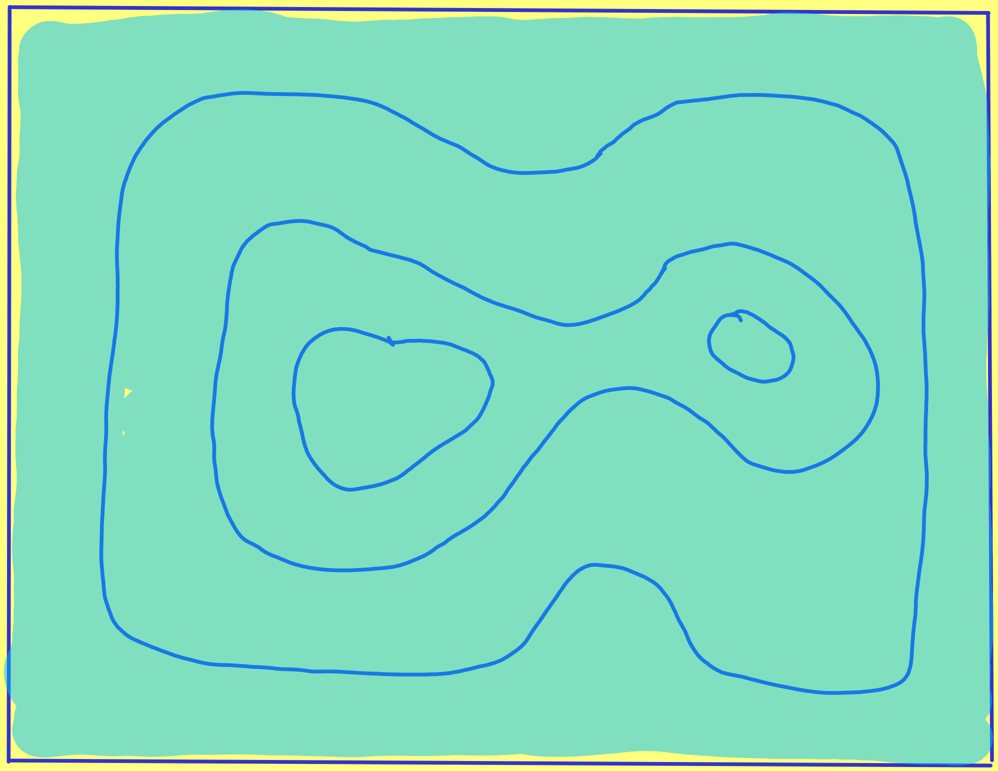

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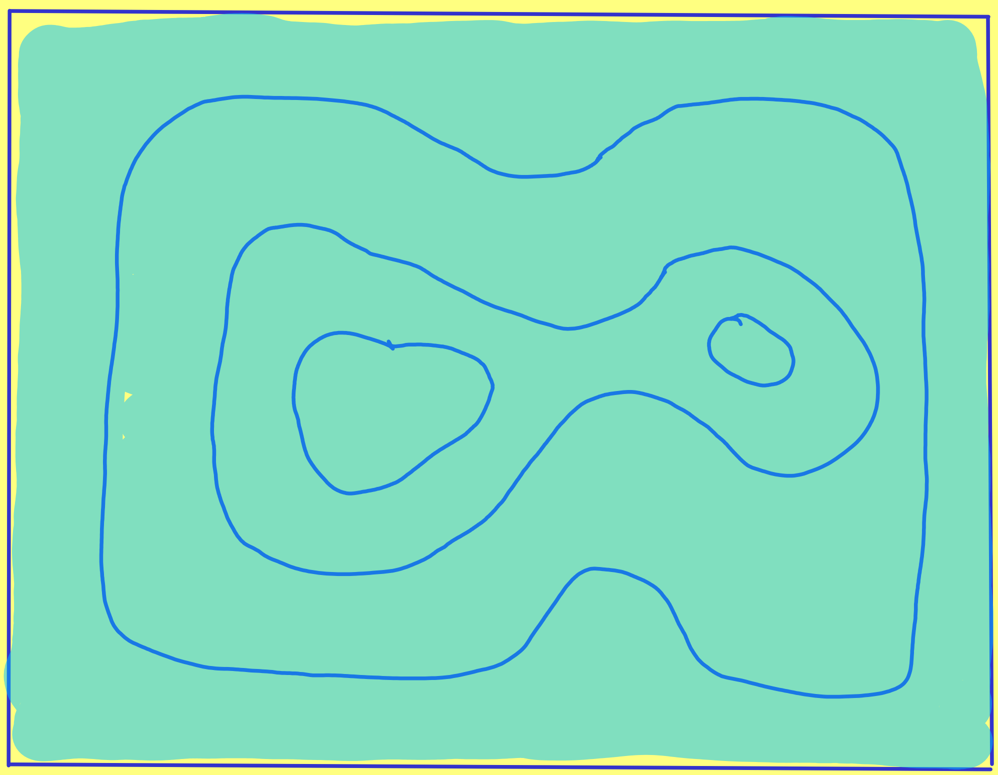

β_0 = #components

β_1 = #holes

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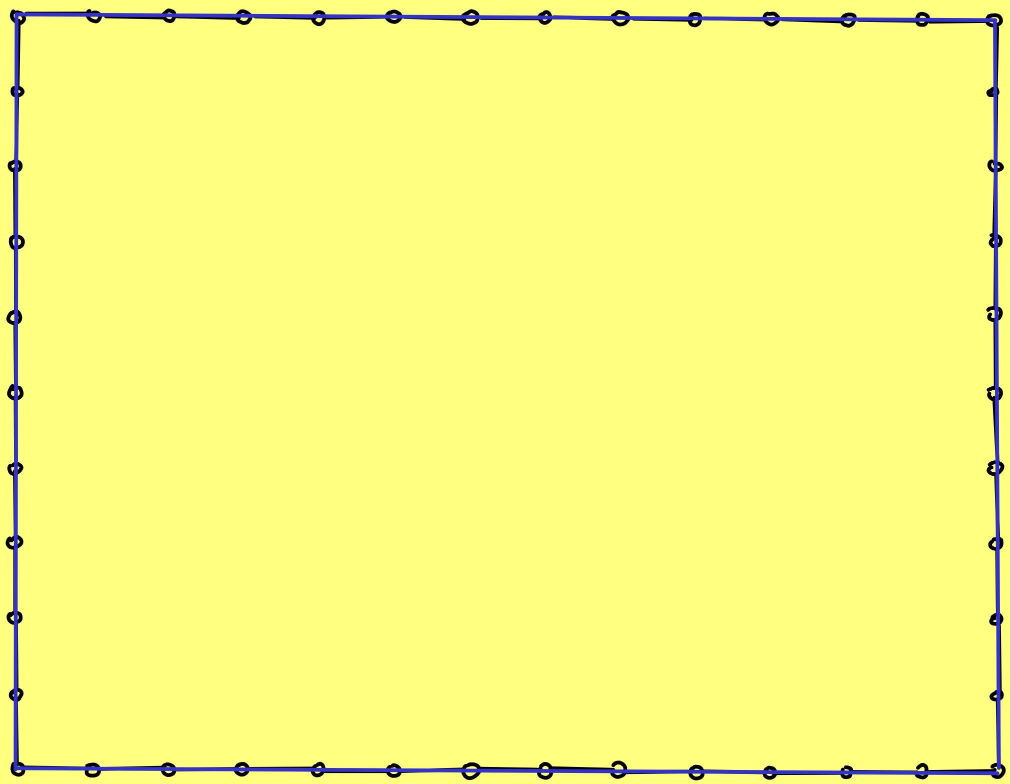
$f : \mathbb{R}^2 \to \mathbb{R}$

induced subcomplexes K_0



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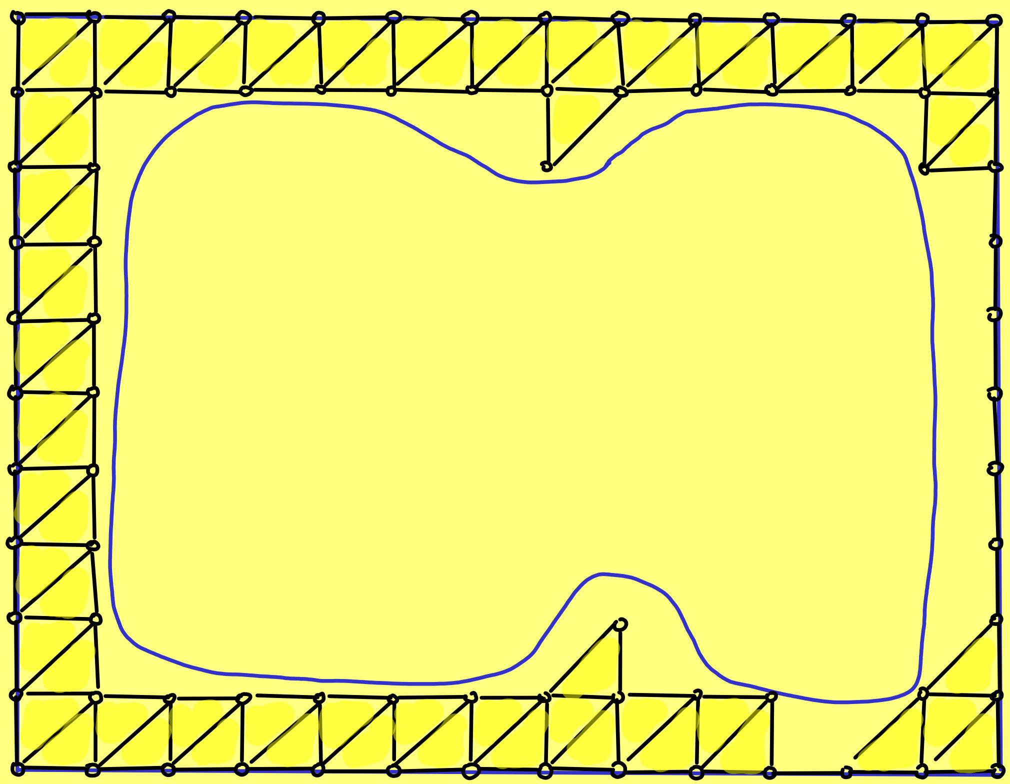
induced subcomplexes $K_0 \subseteq K_1$



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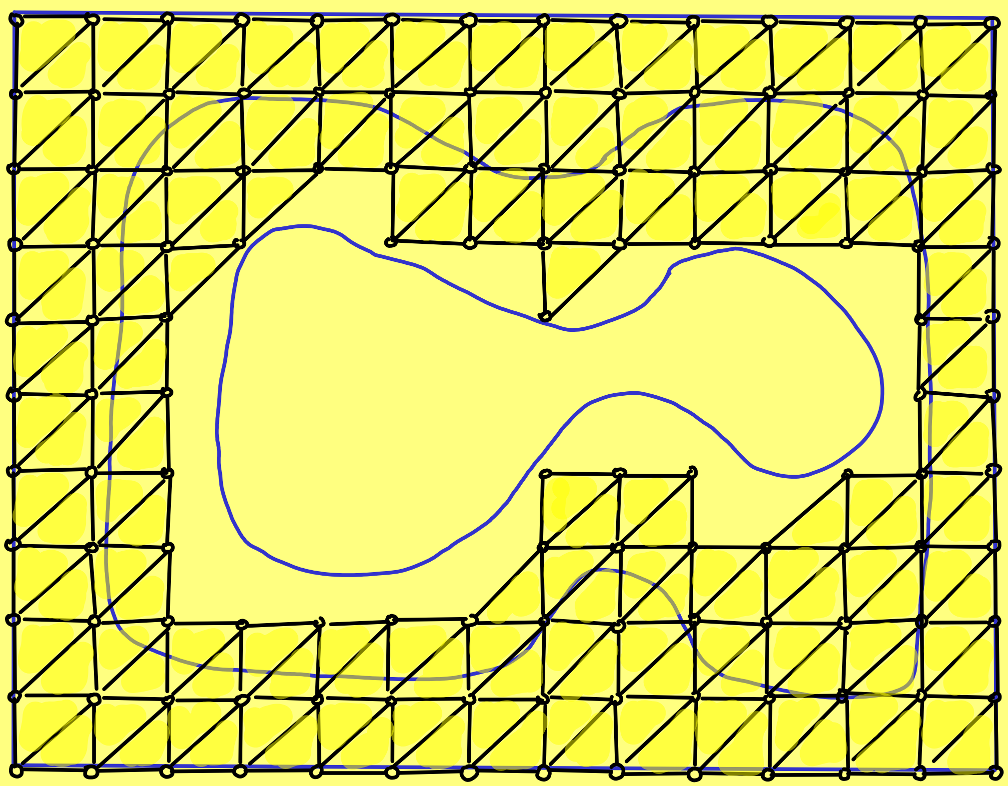
induced subcomplexes $K_0 \subseteq K_1 \subseteq K_2$



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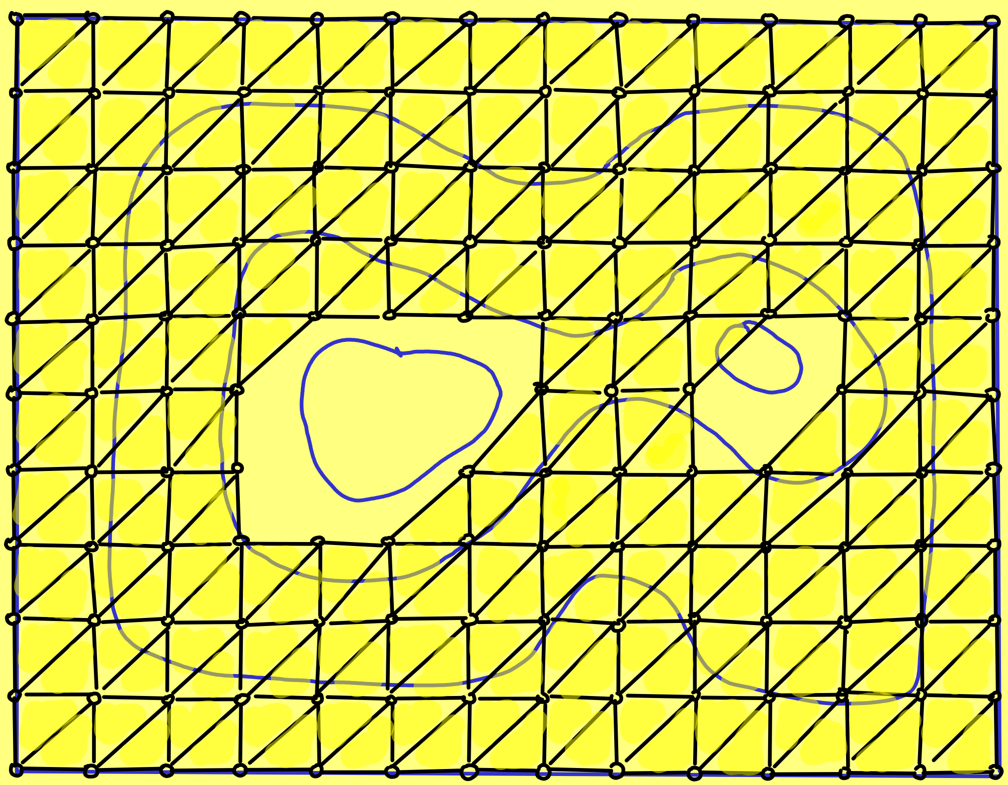
induced subcomplexes $K_0 \subseteq K_1 \subseteq K_2 \subseteq K_3$



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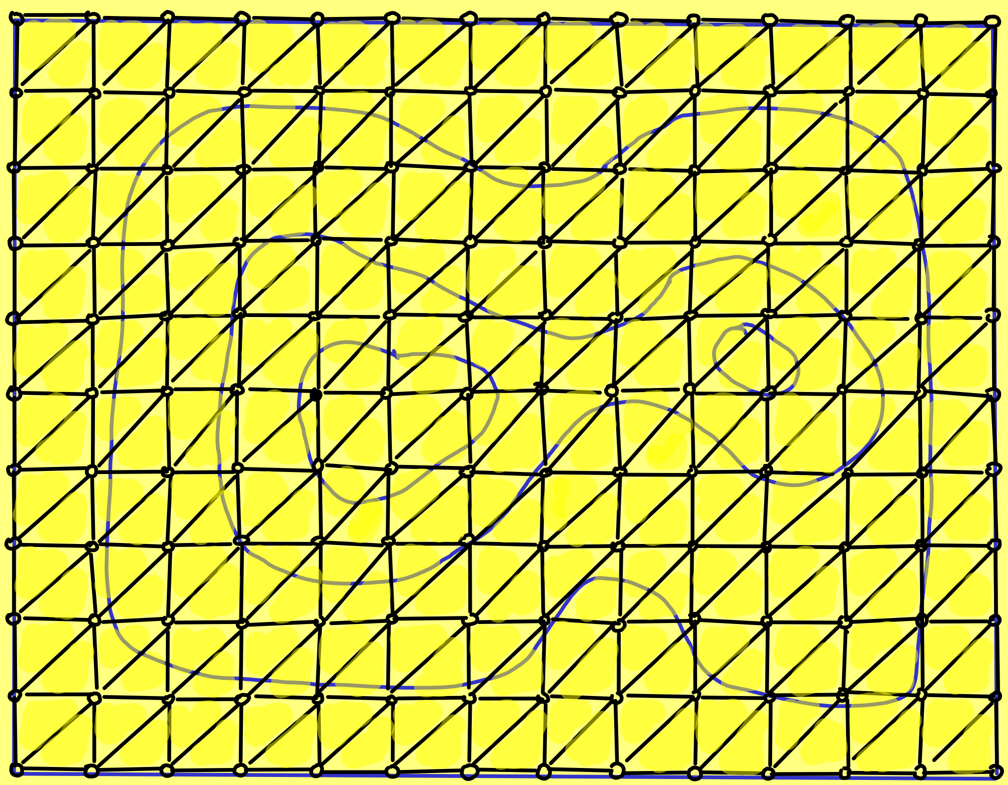

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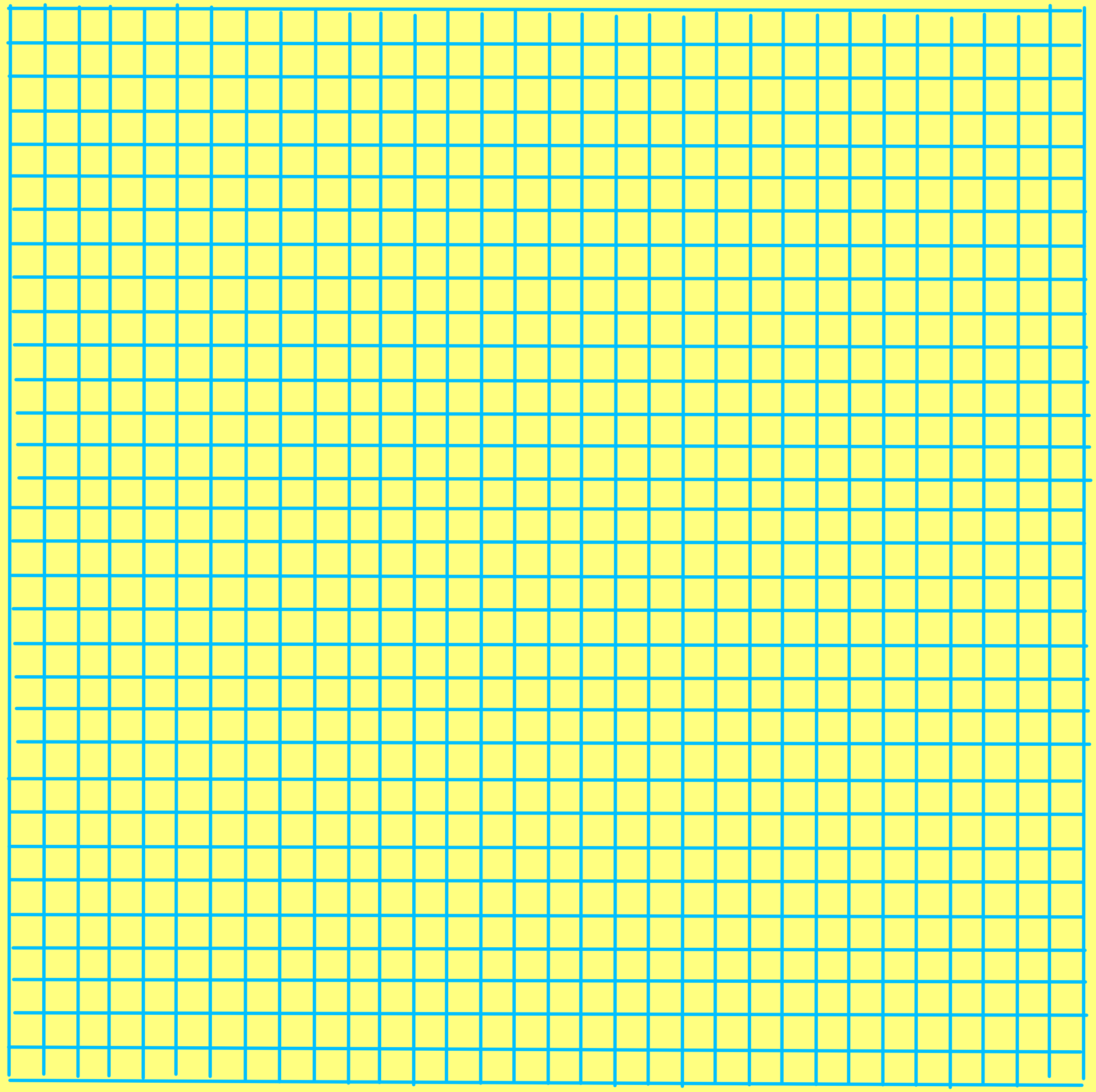
I.3 Quadtree



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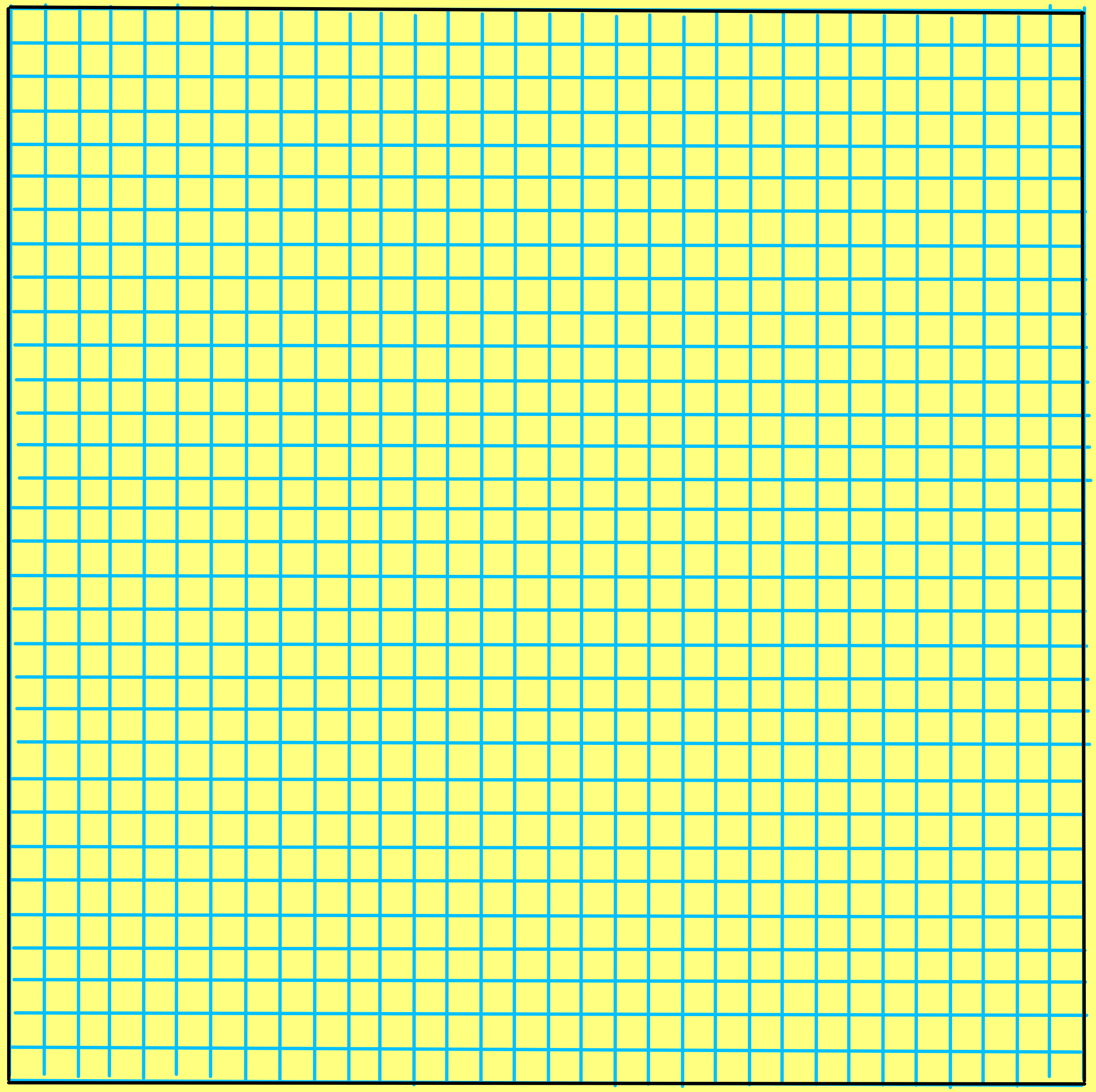
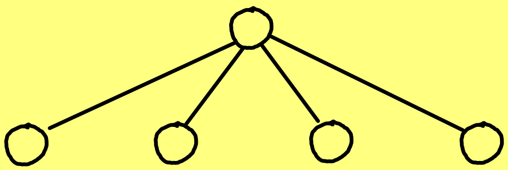
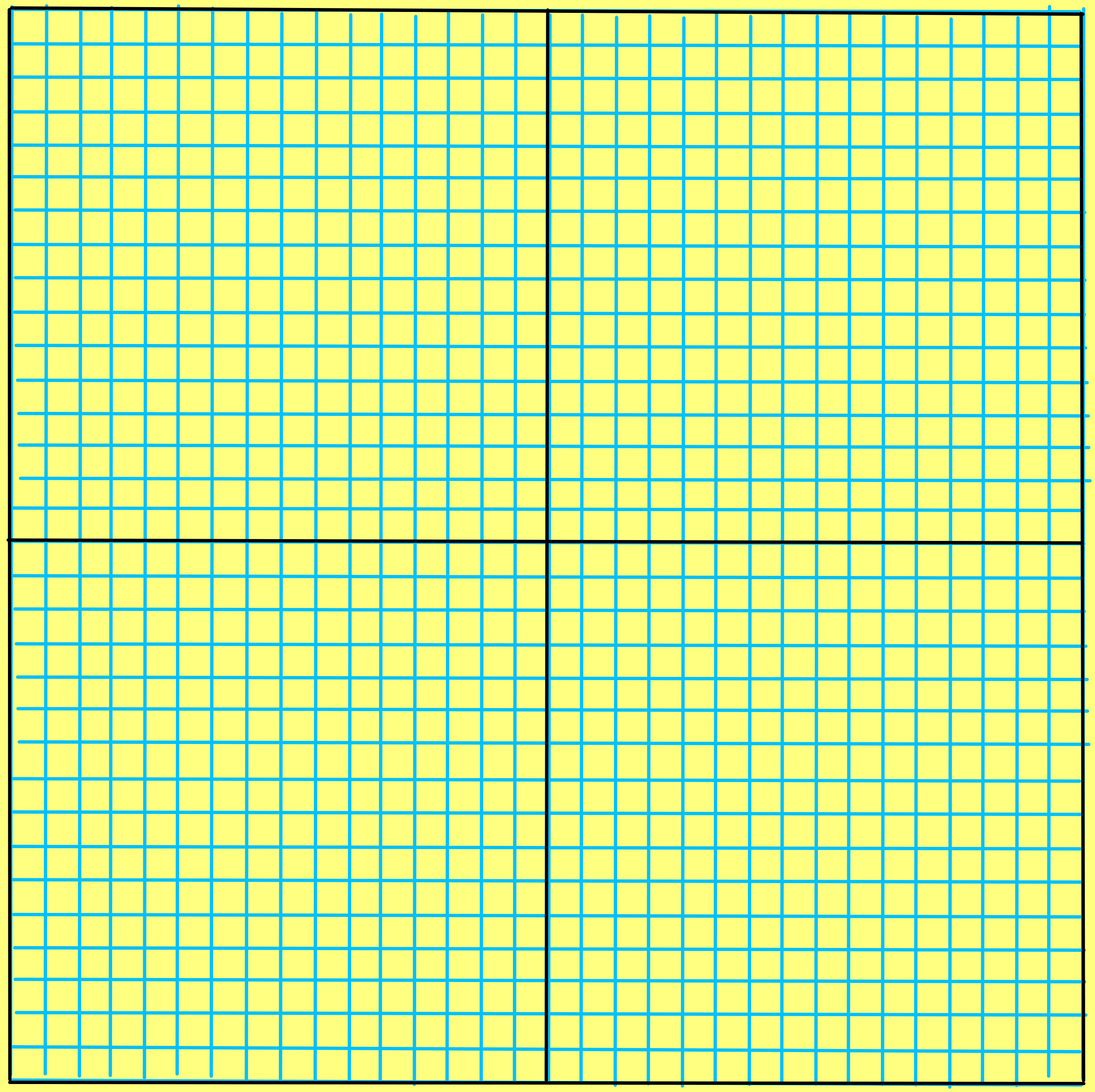


I.3 QUADTREE





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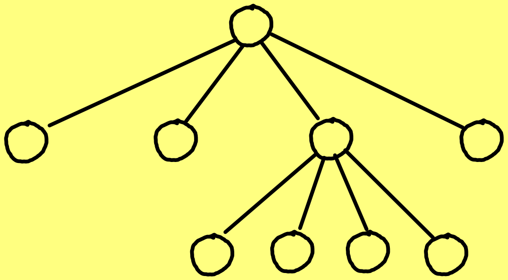

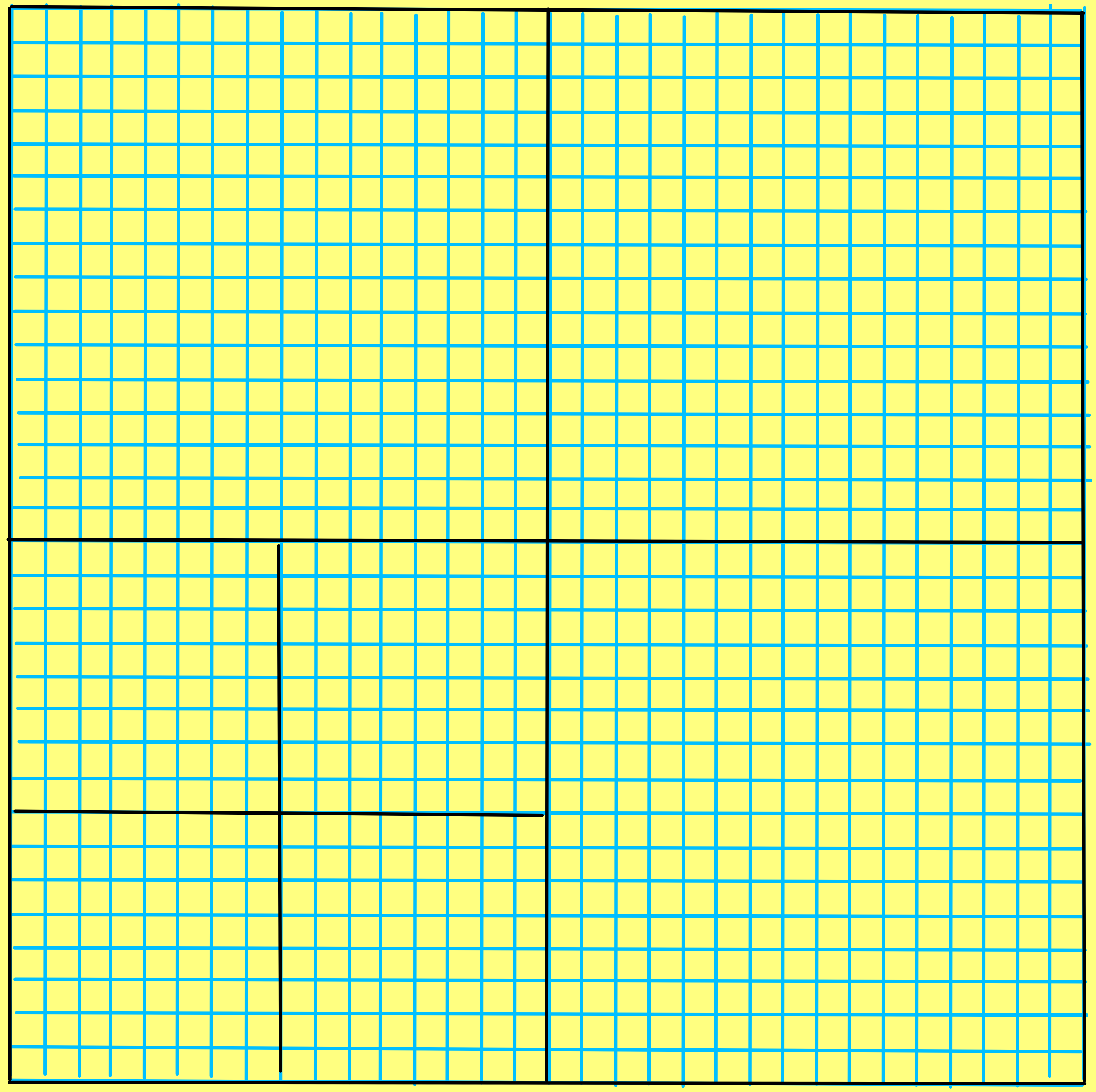

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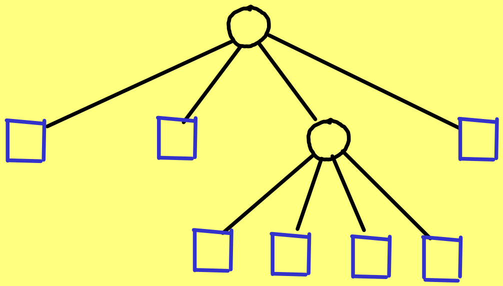

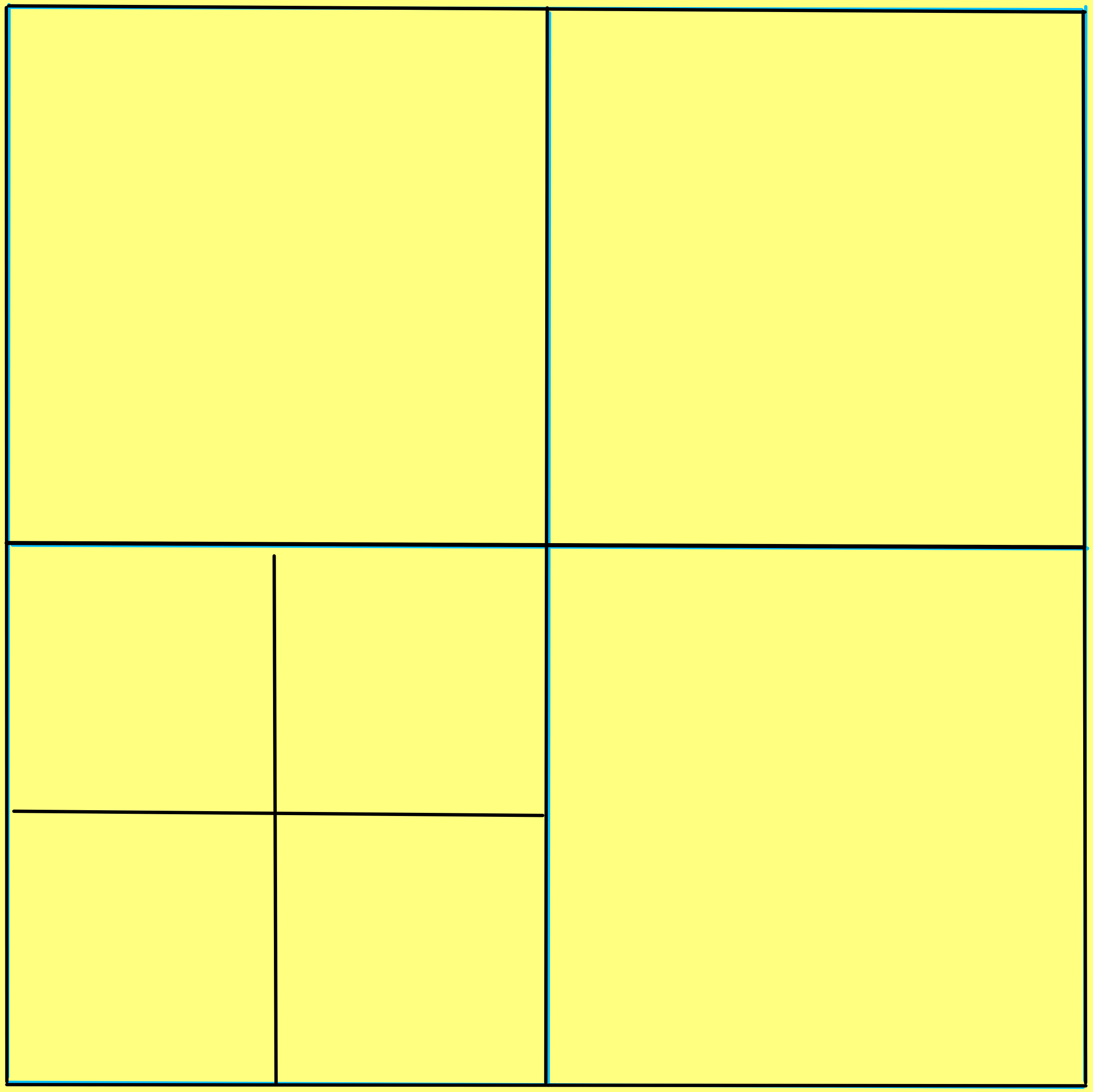

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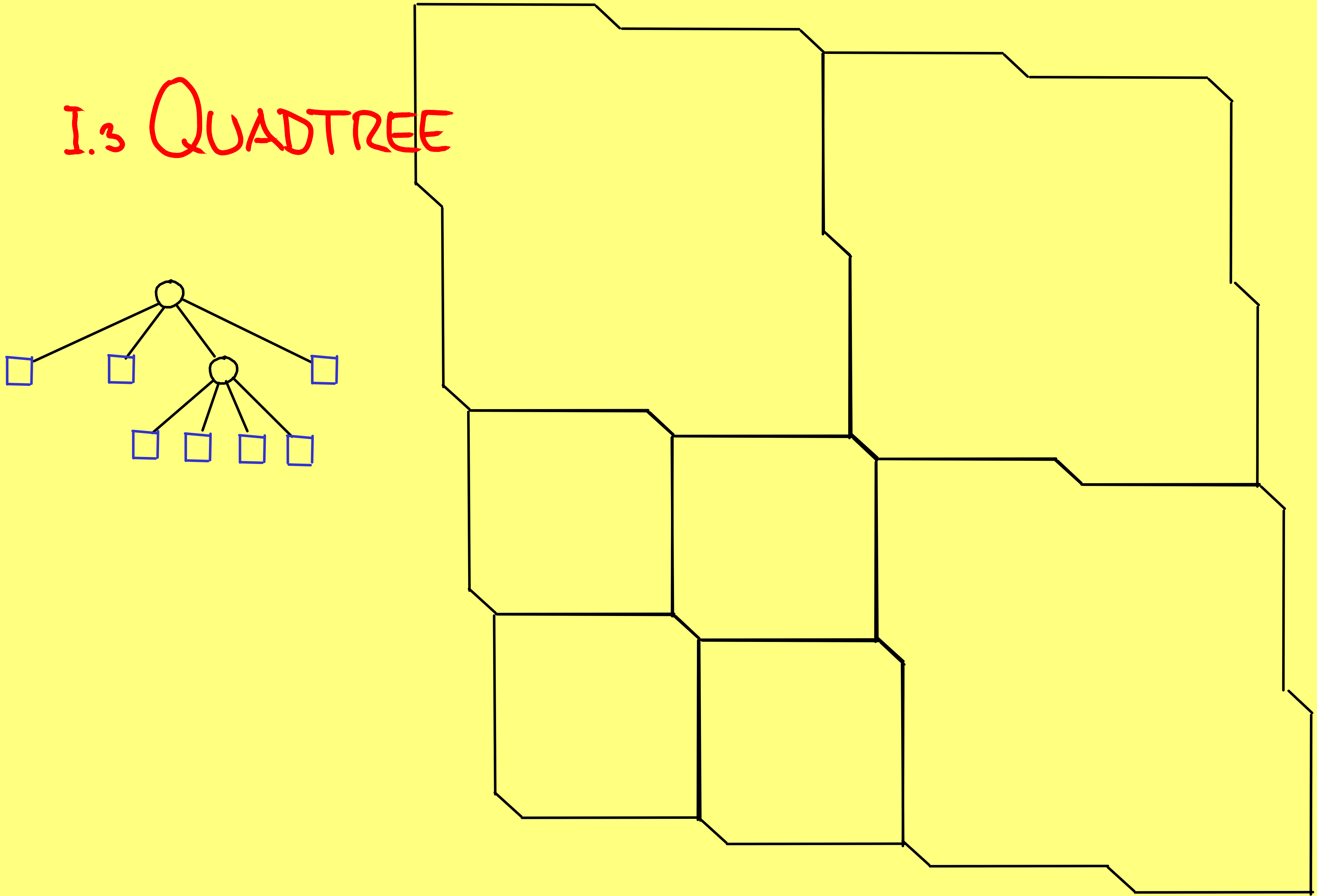

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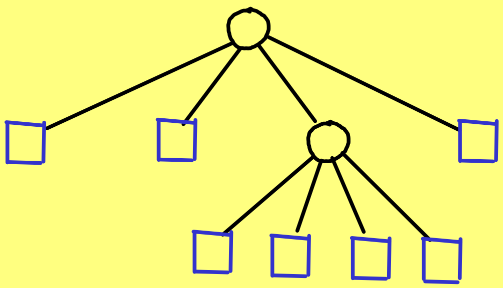

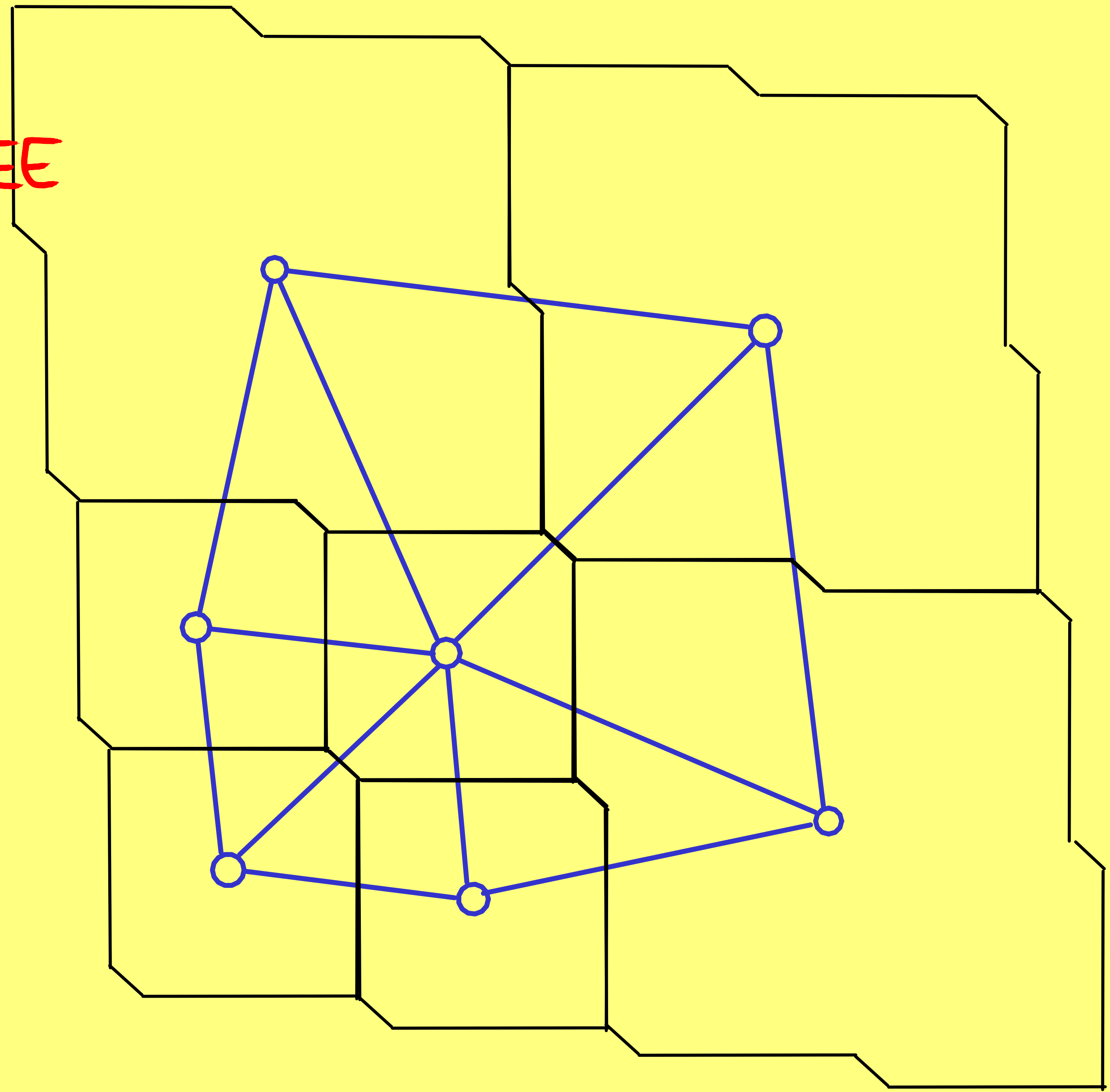

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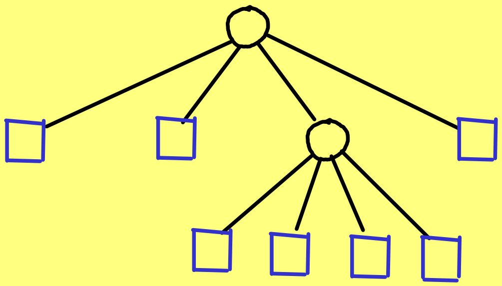

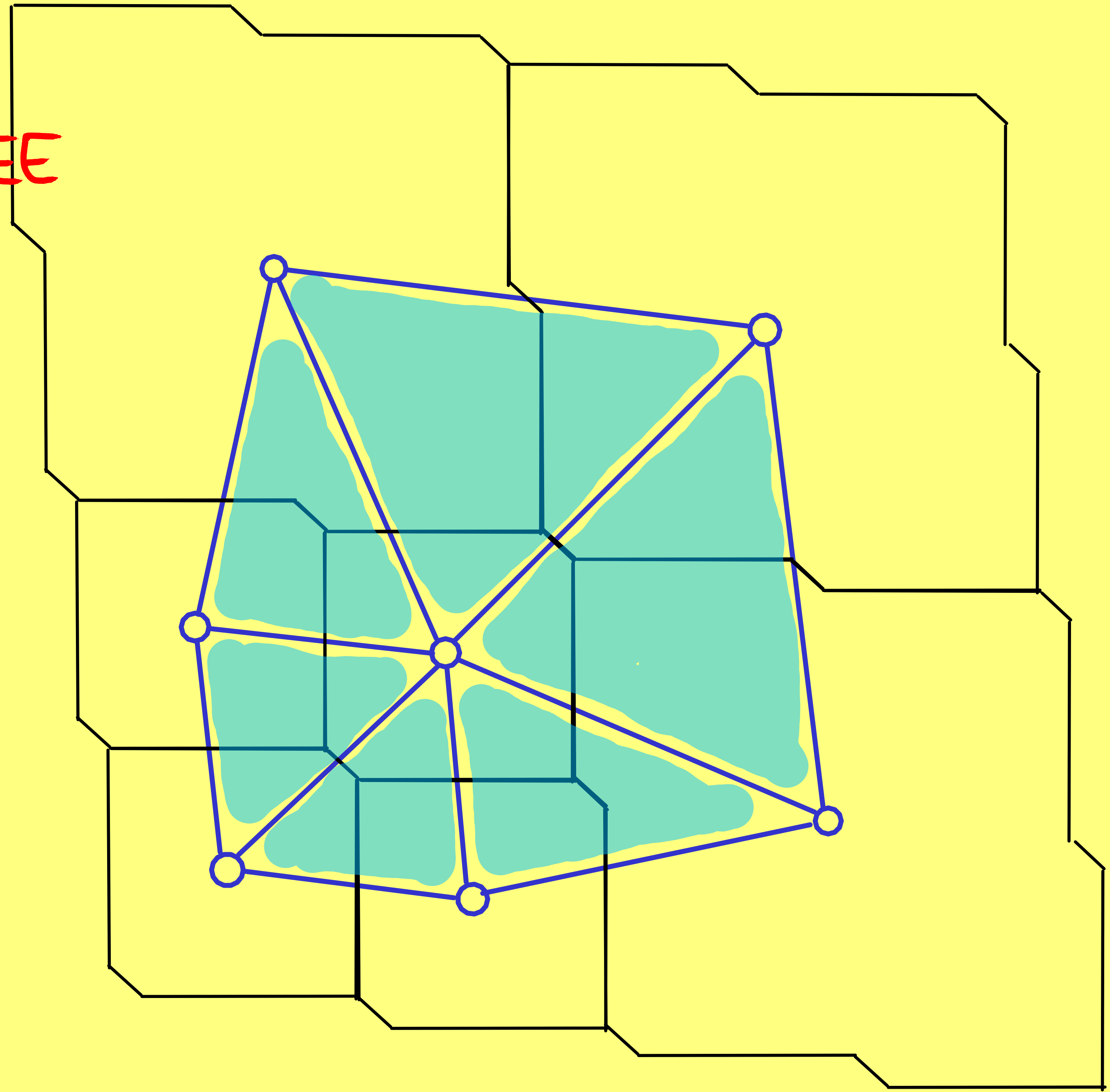

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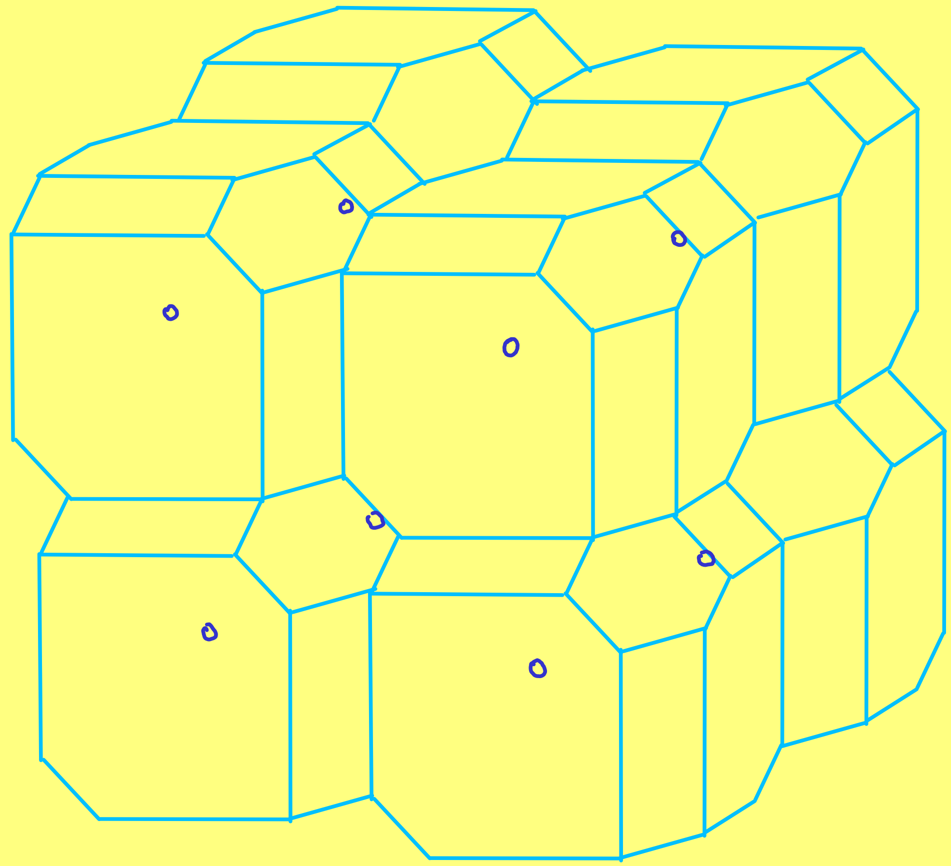

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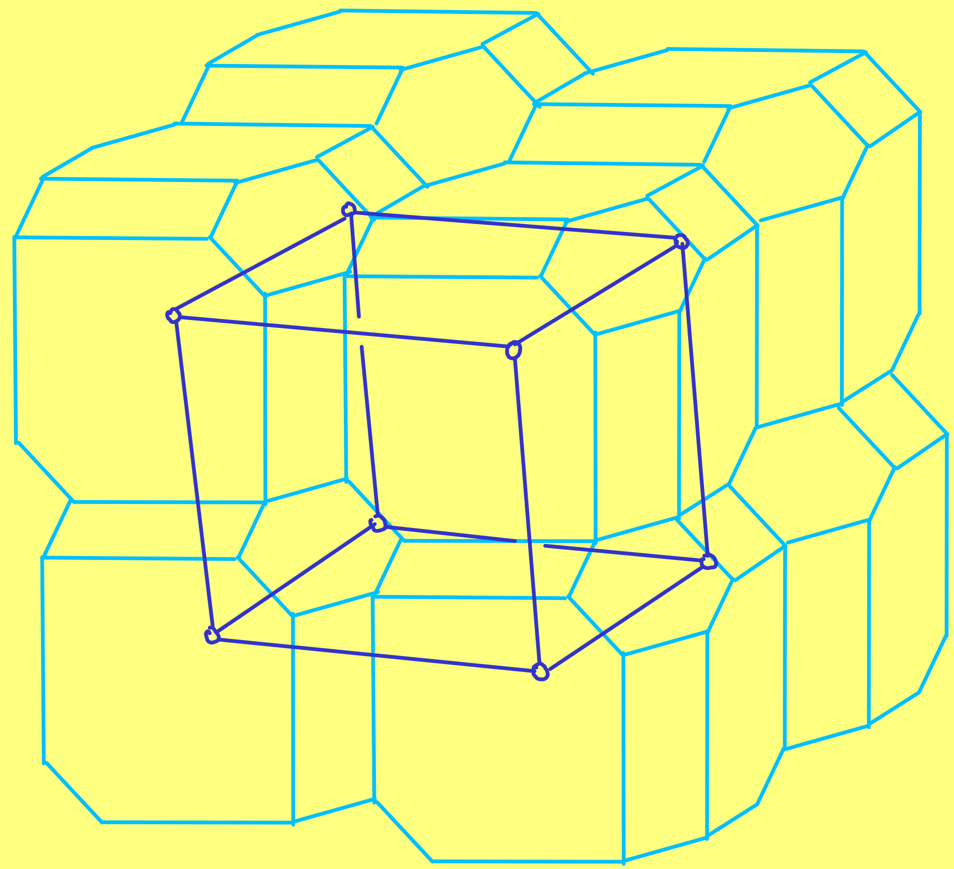

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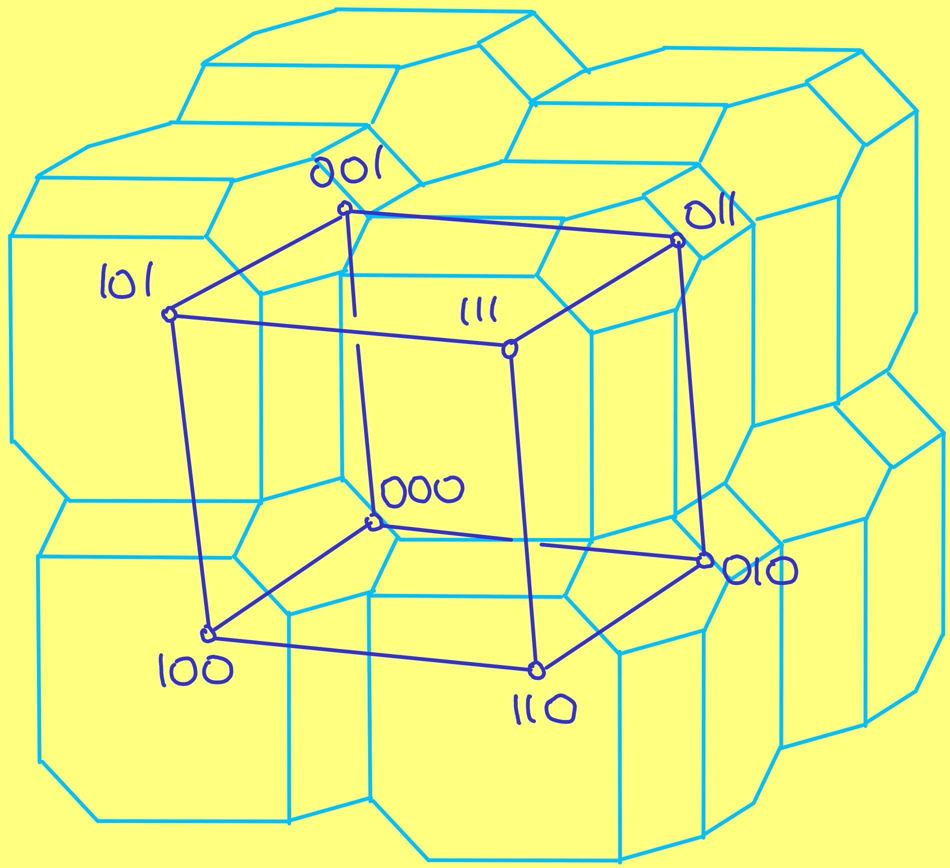

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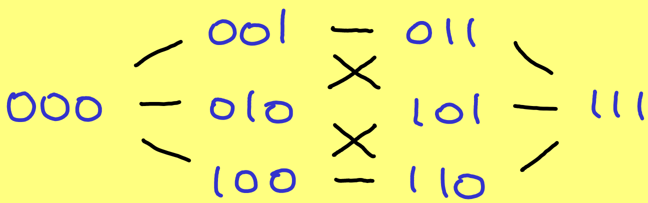

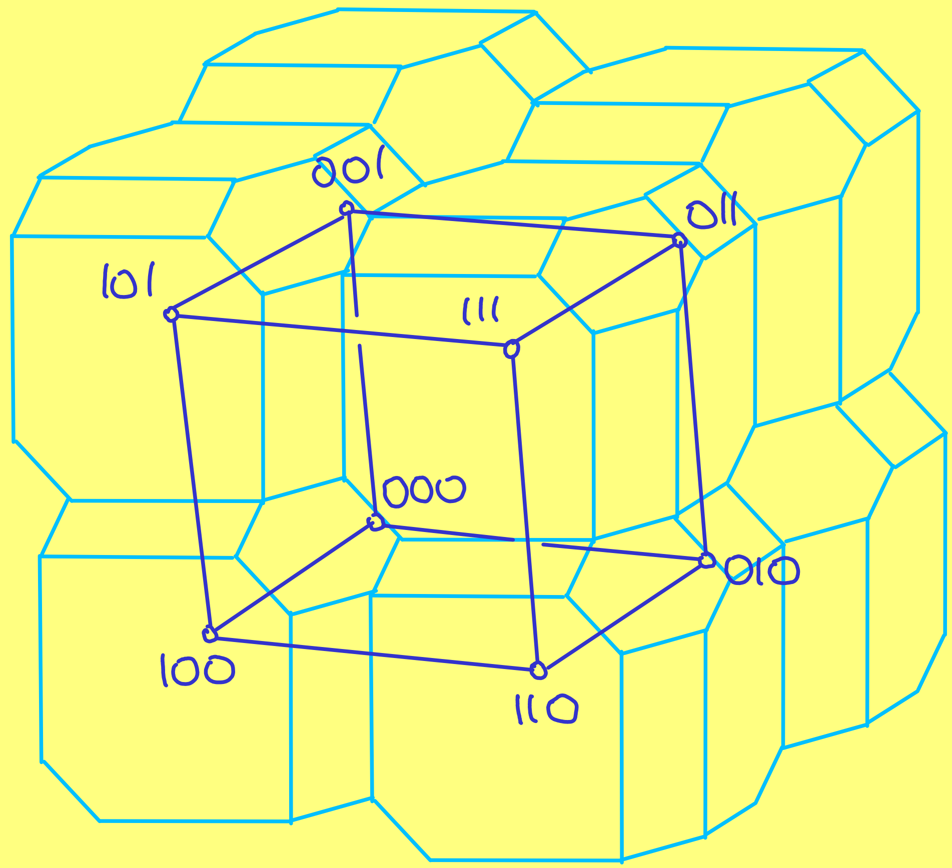

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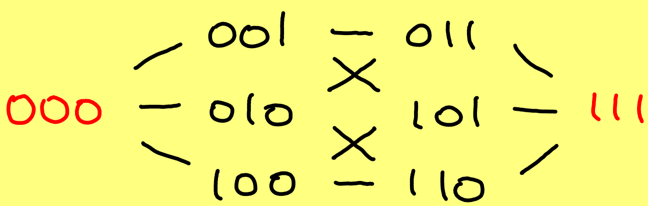

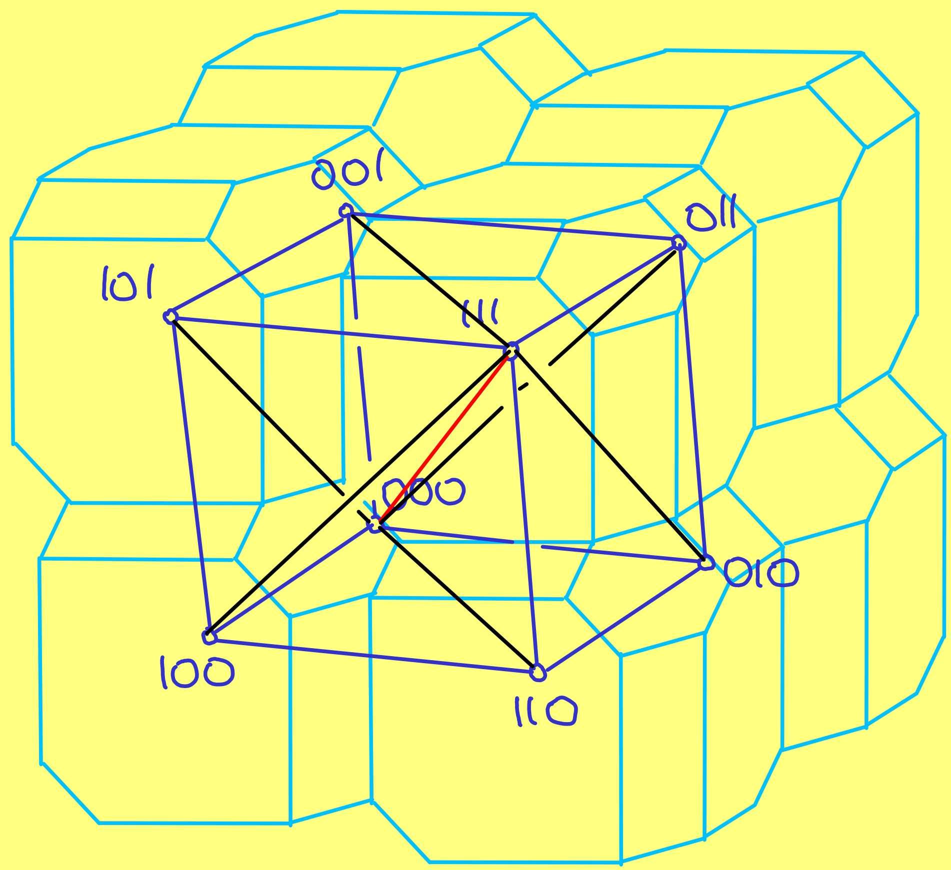

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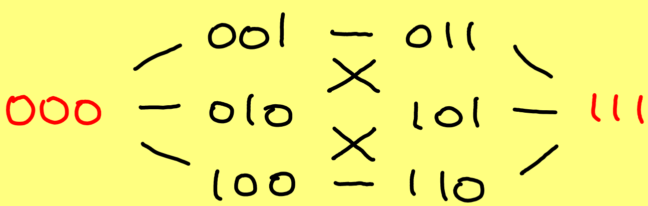
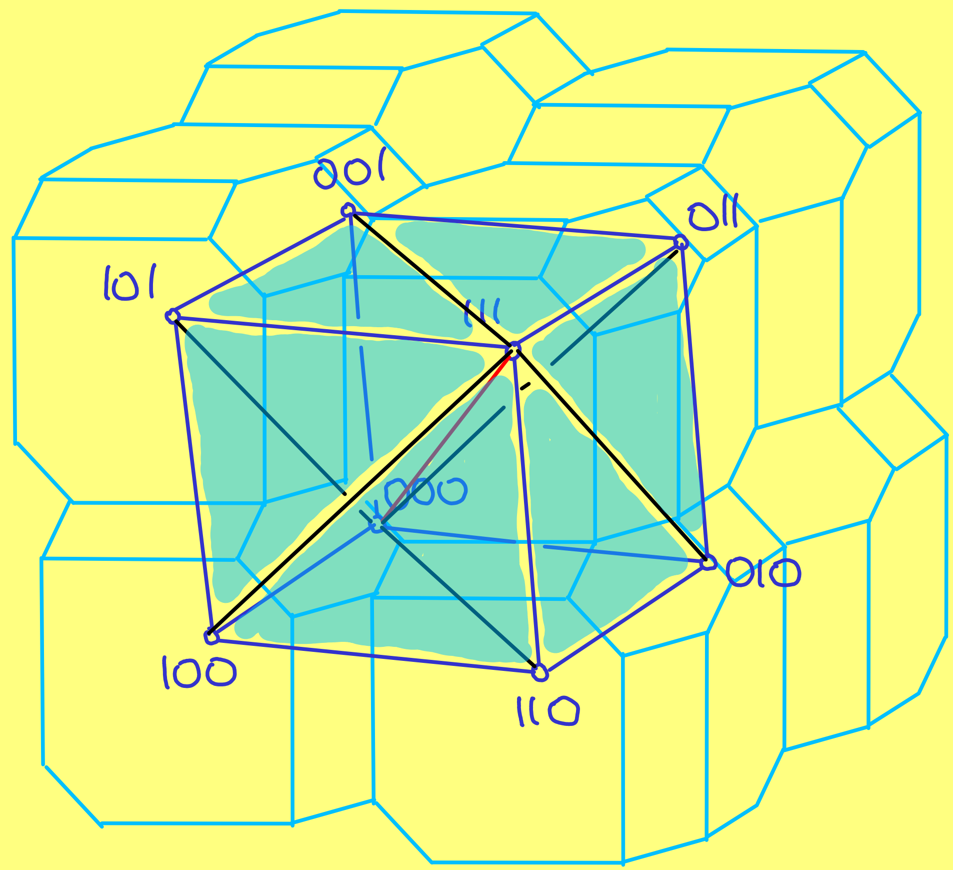


Freudenthal tri.:

simplices are convex hulls of chains



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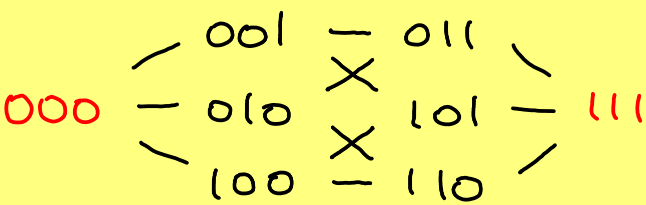

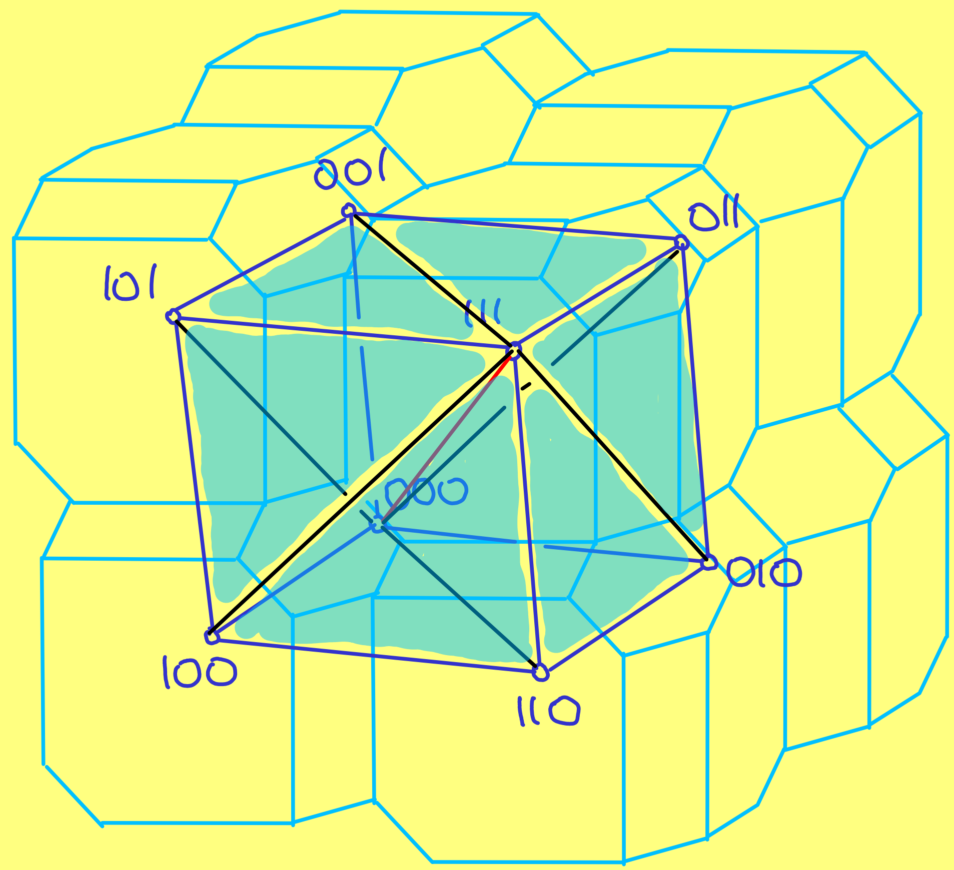

Freudenthal tri.:

simplices are convex hulls of chains

Thm. Octtree balanced $\Rightarrow$ dual complex embedded.

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Moments IV Scale Space

II.1 Absolute Homology

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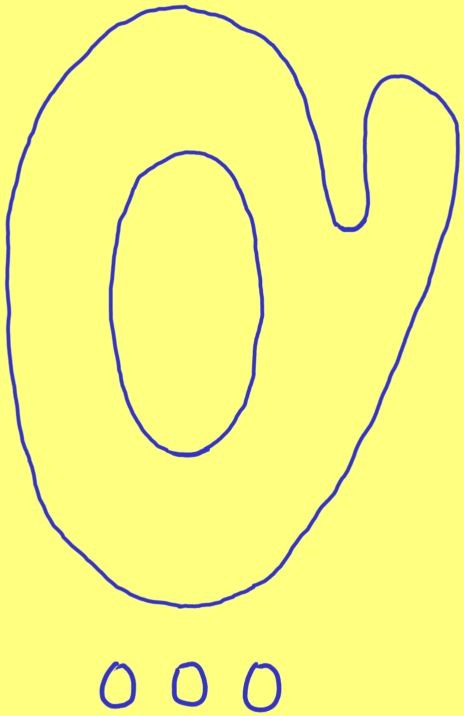

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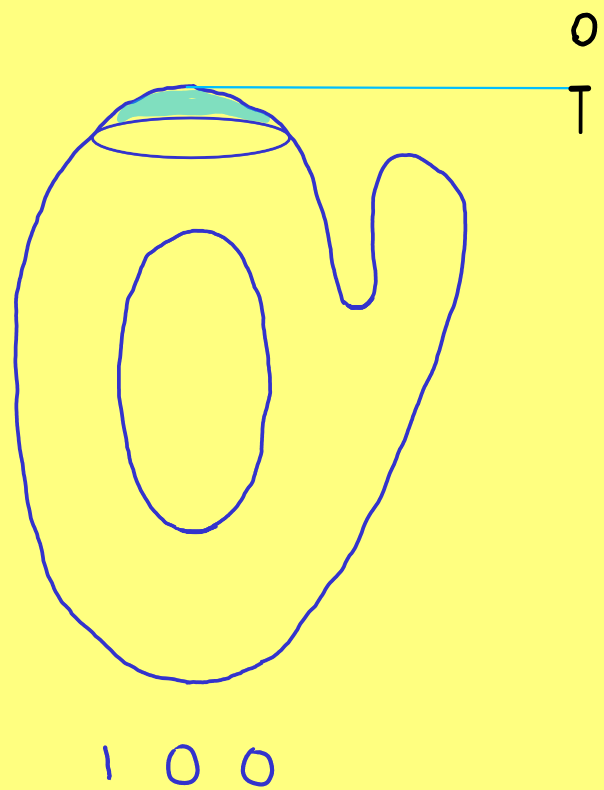

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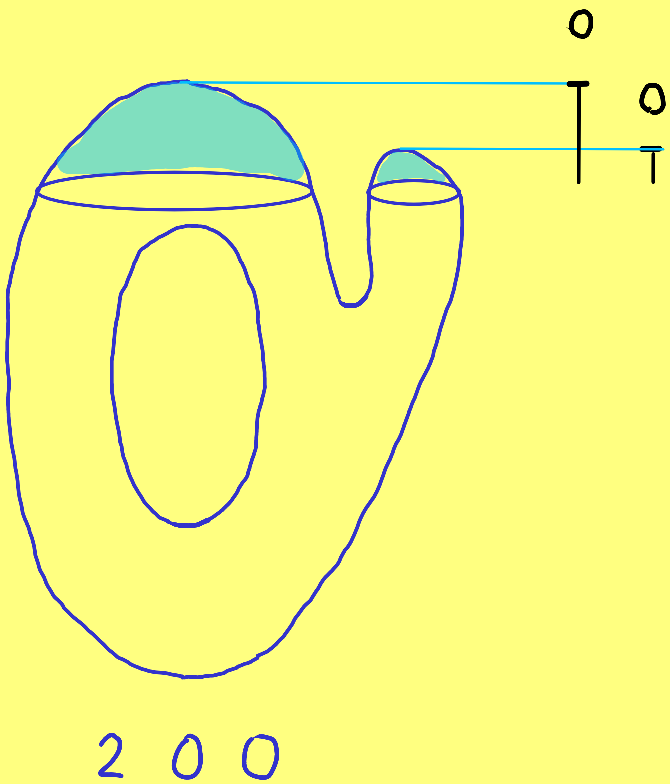

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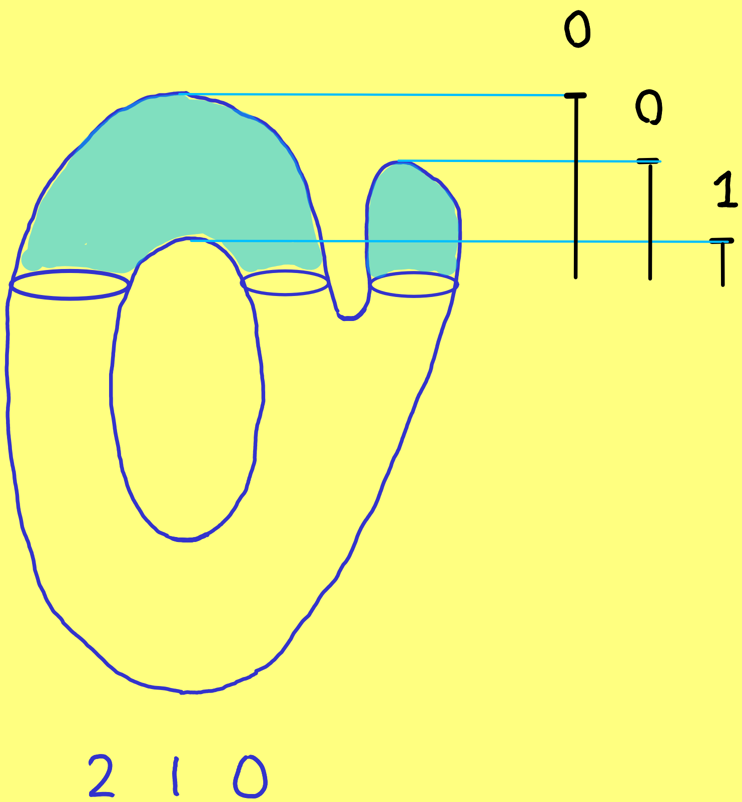

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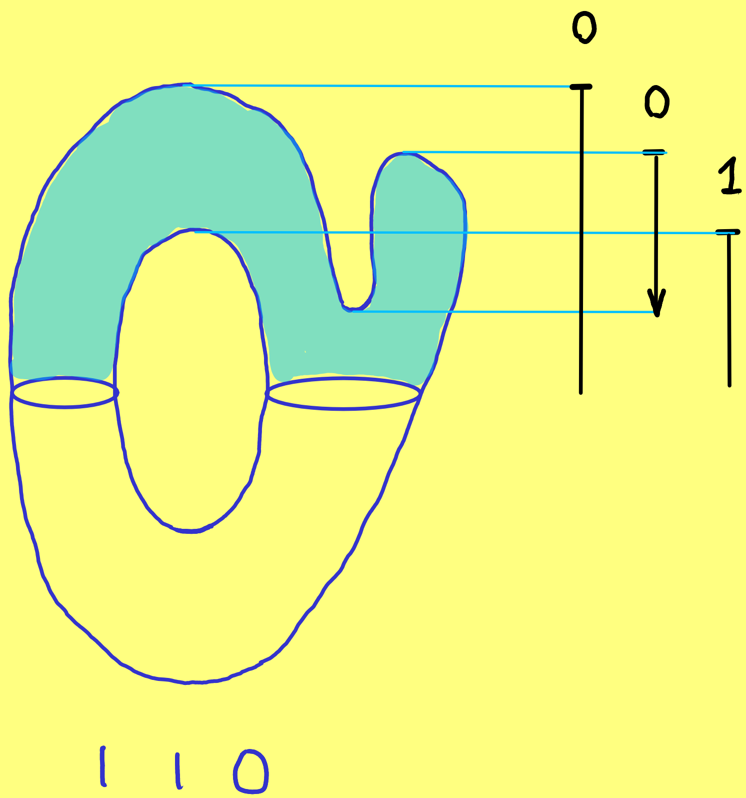

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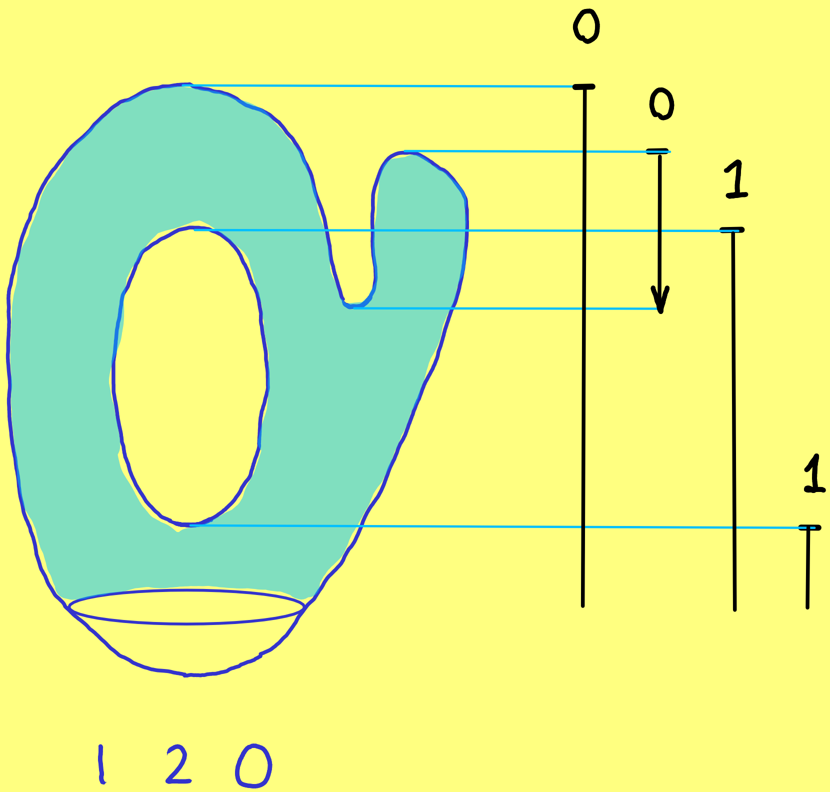

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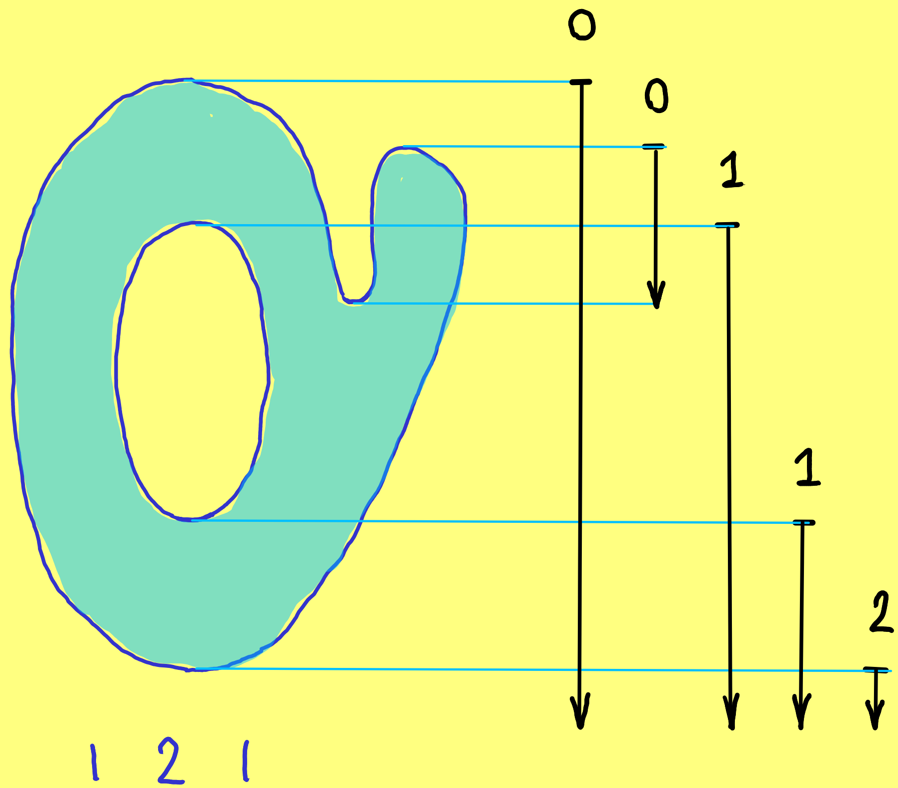
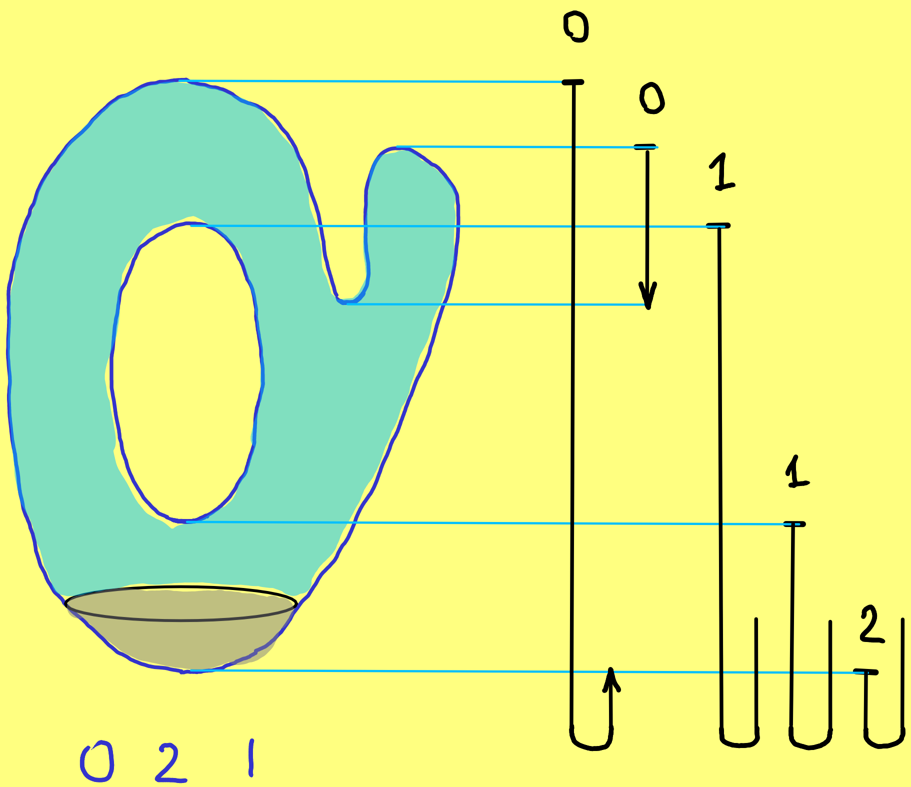


II.2 Relative Homology



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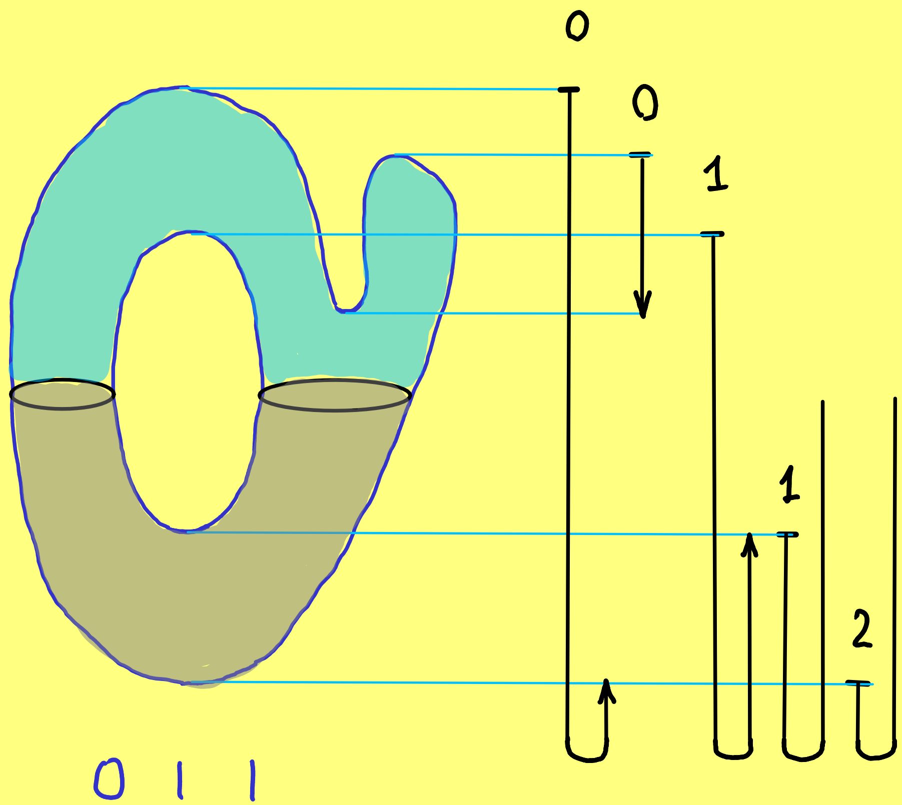

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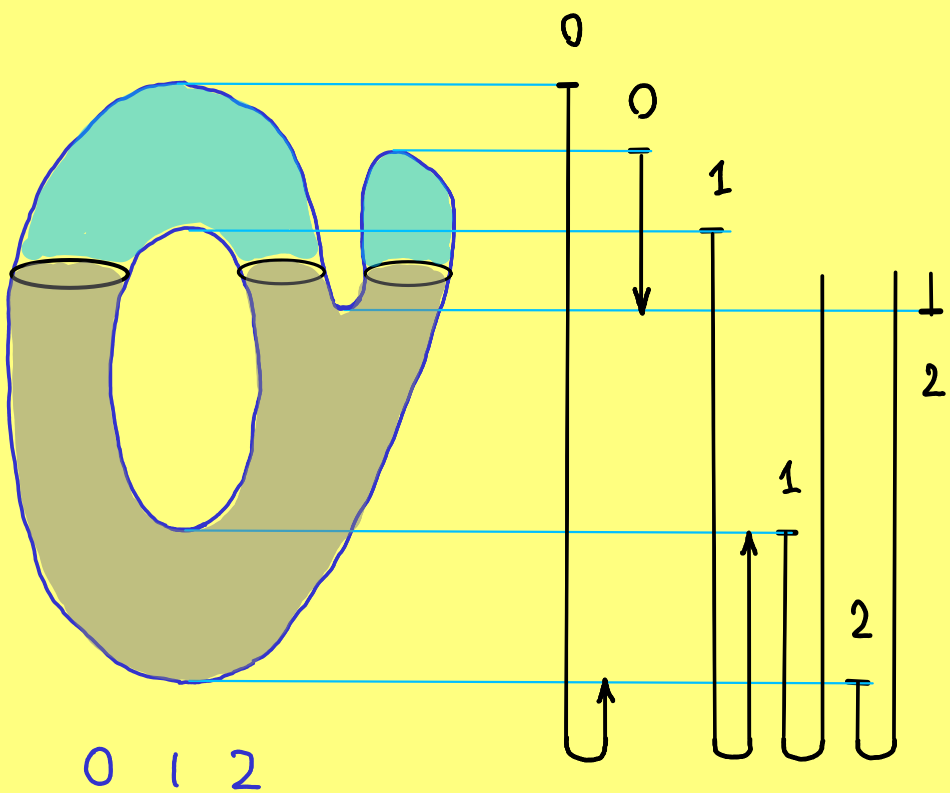

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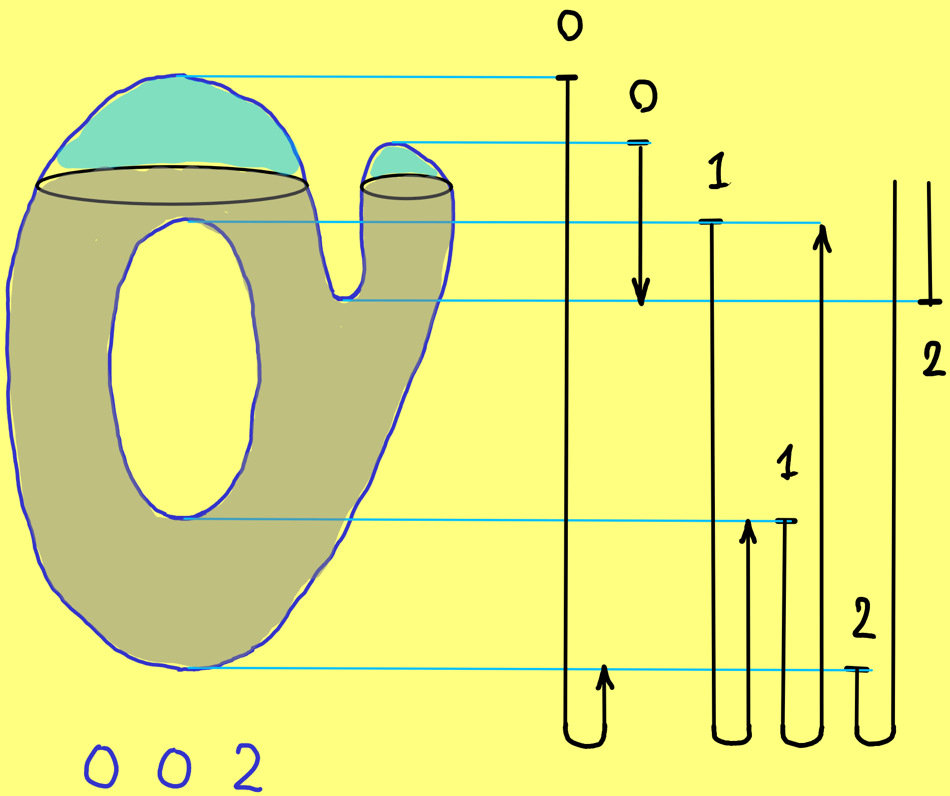

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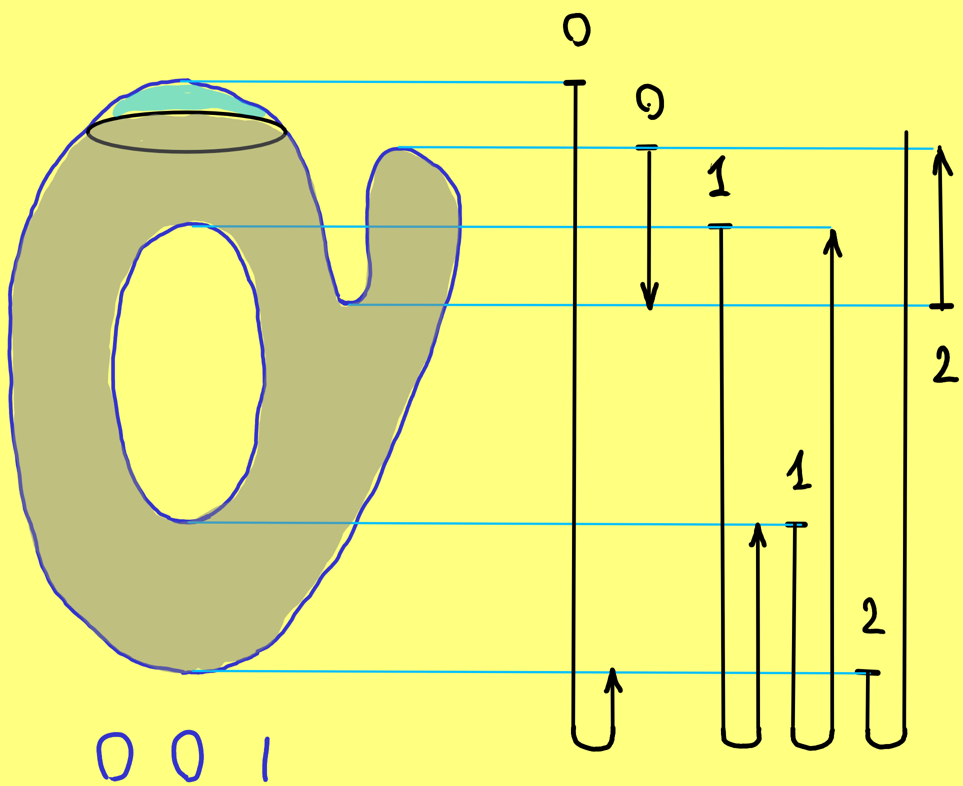
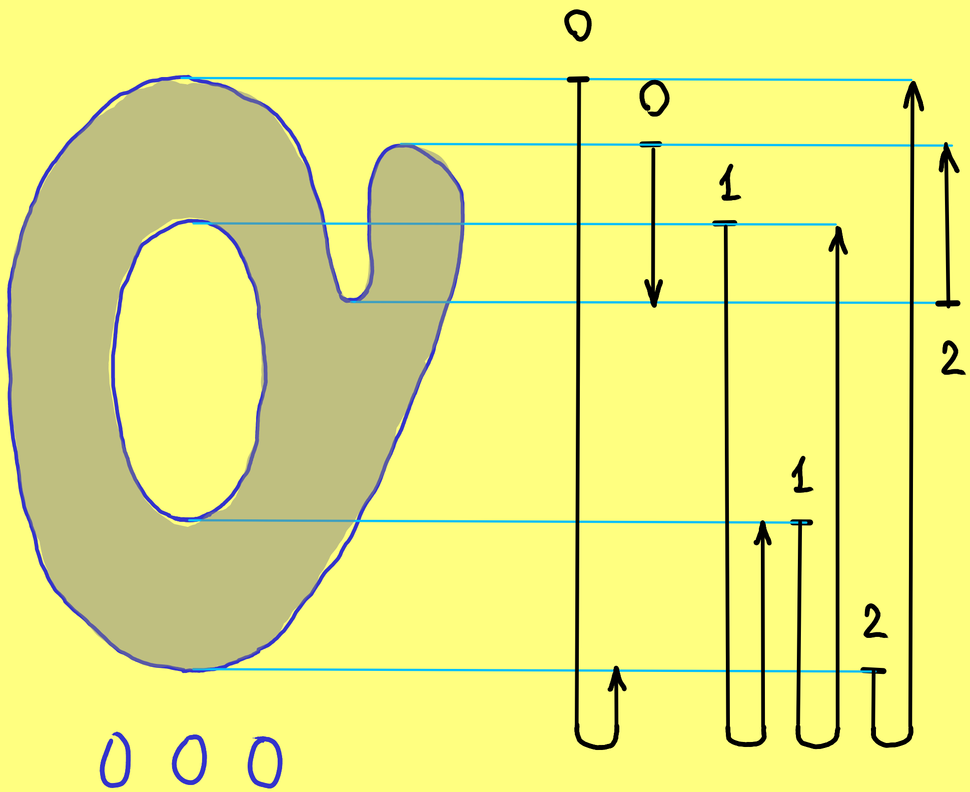
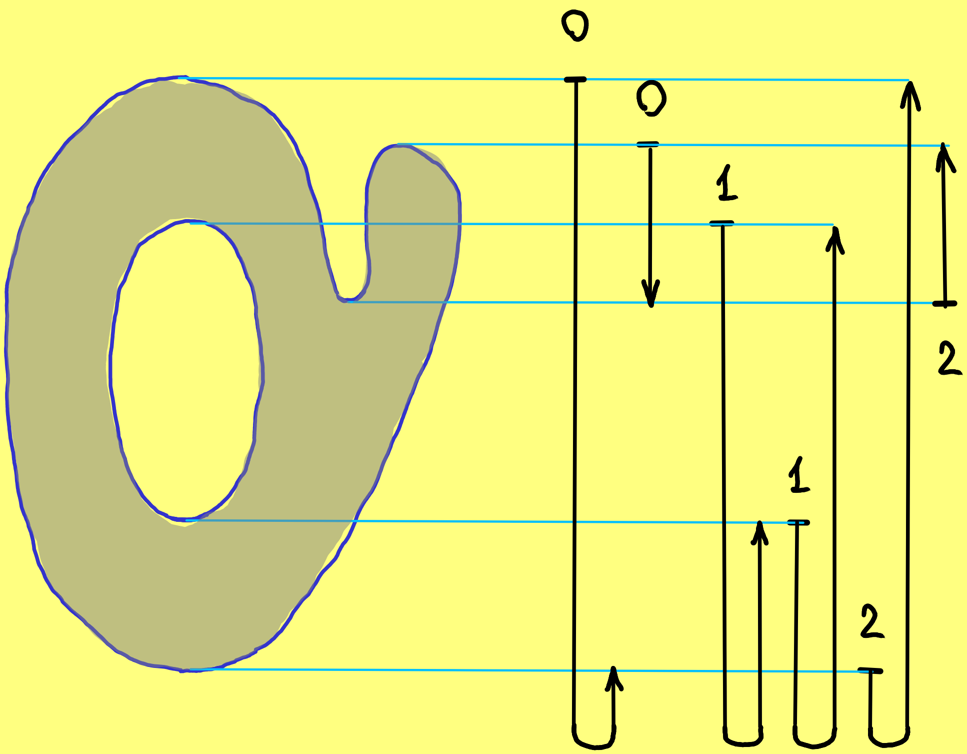
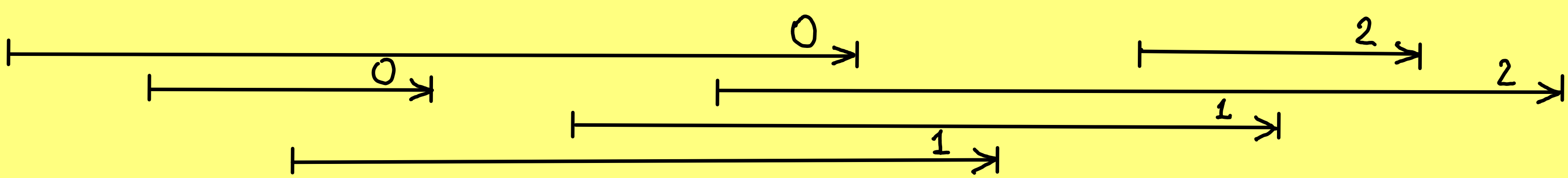


II.2 RELATIVE HOMOLOGY



II.3 Persistence Diagram





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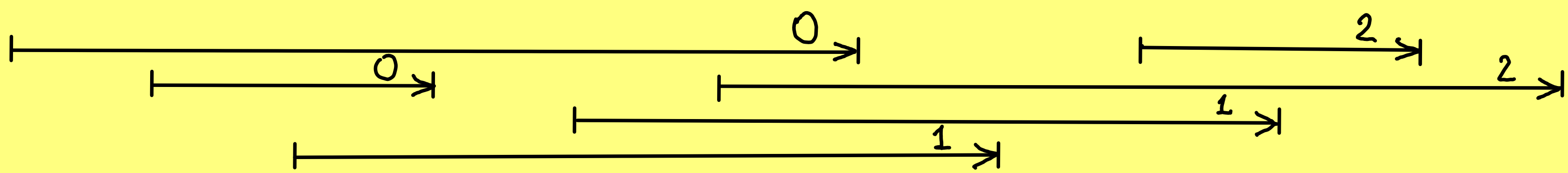

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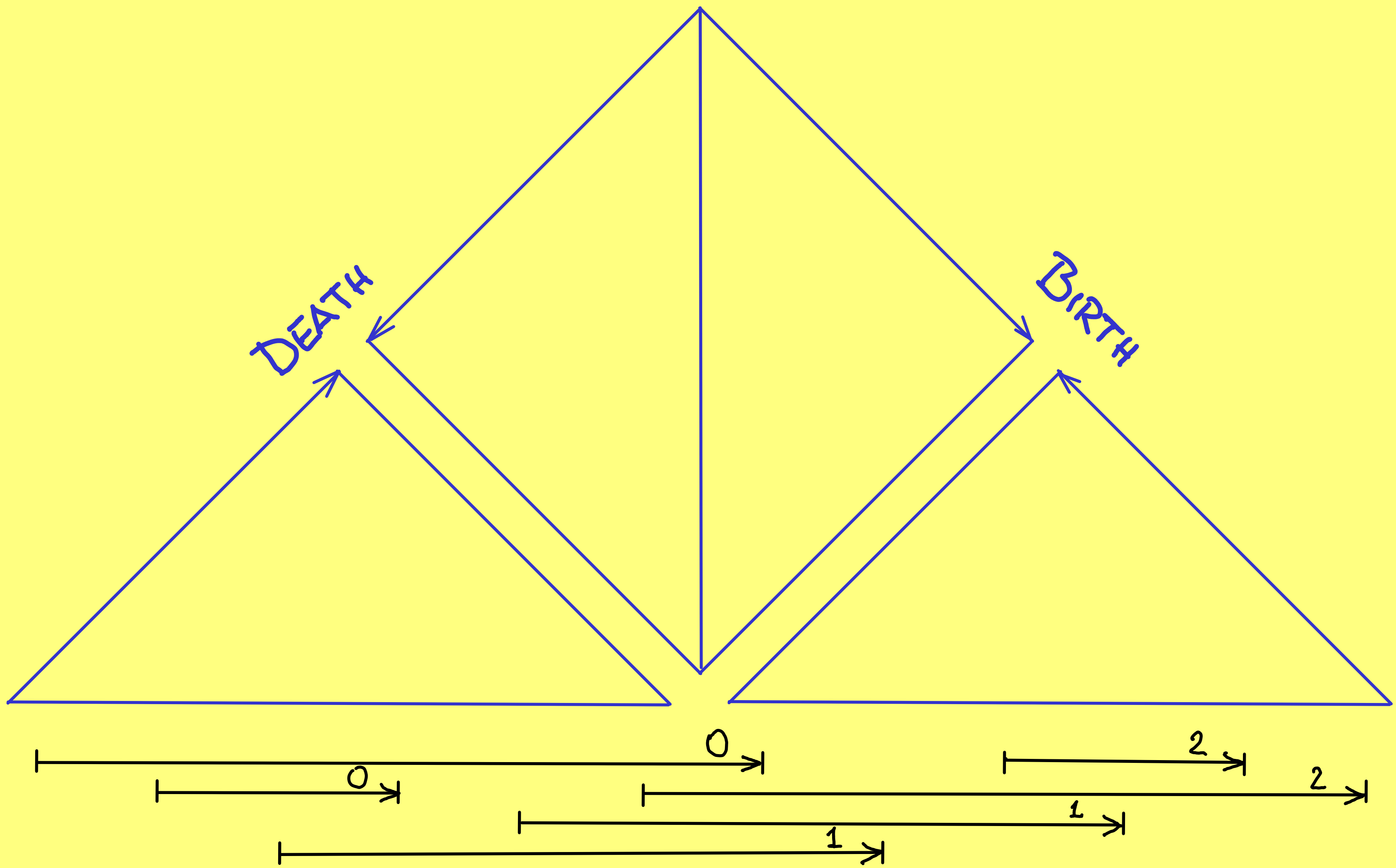

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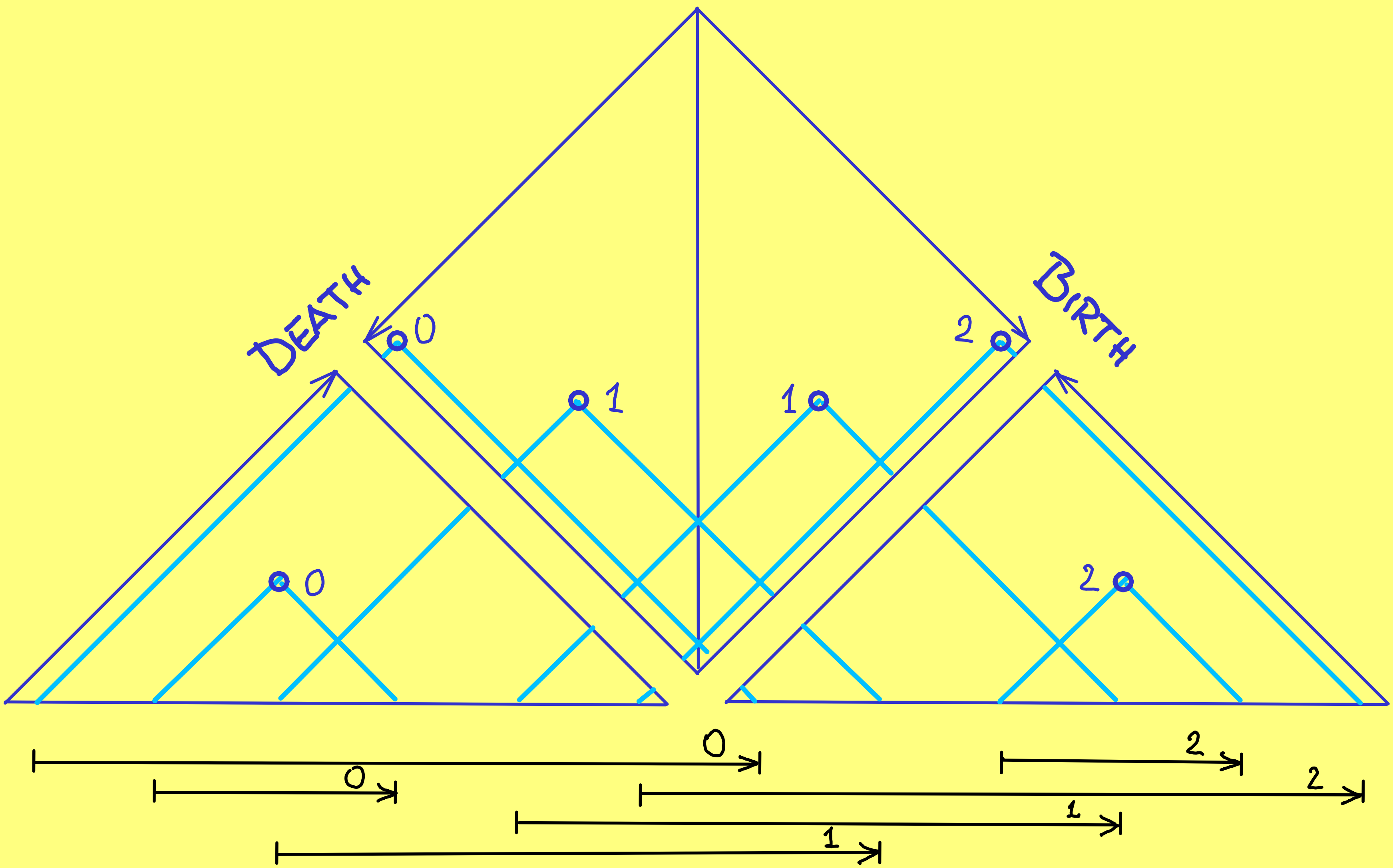

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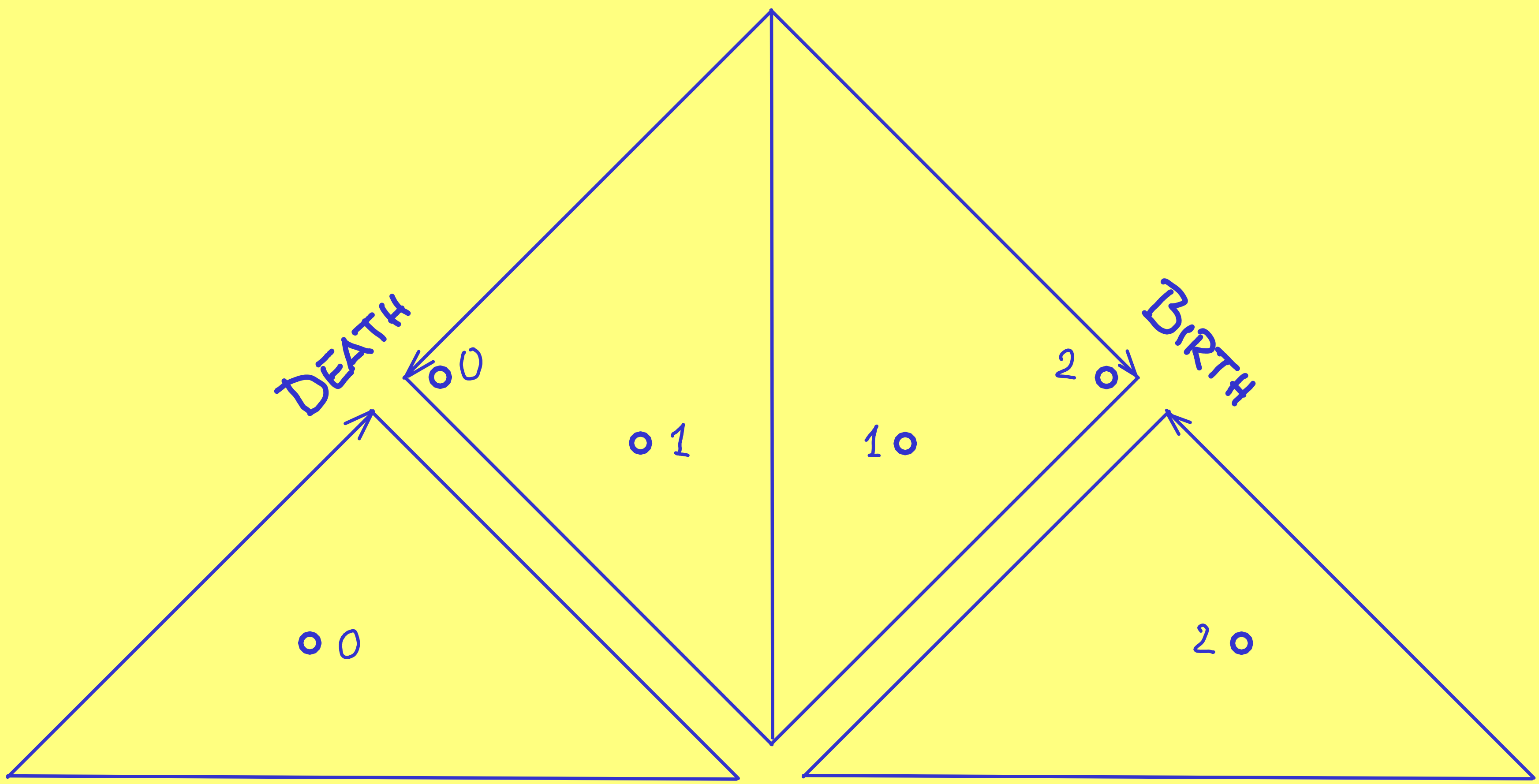

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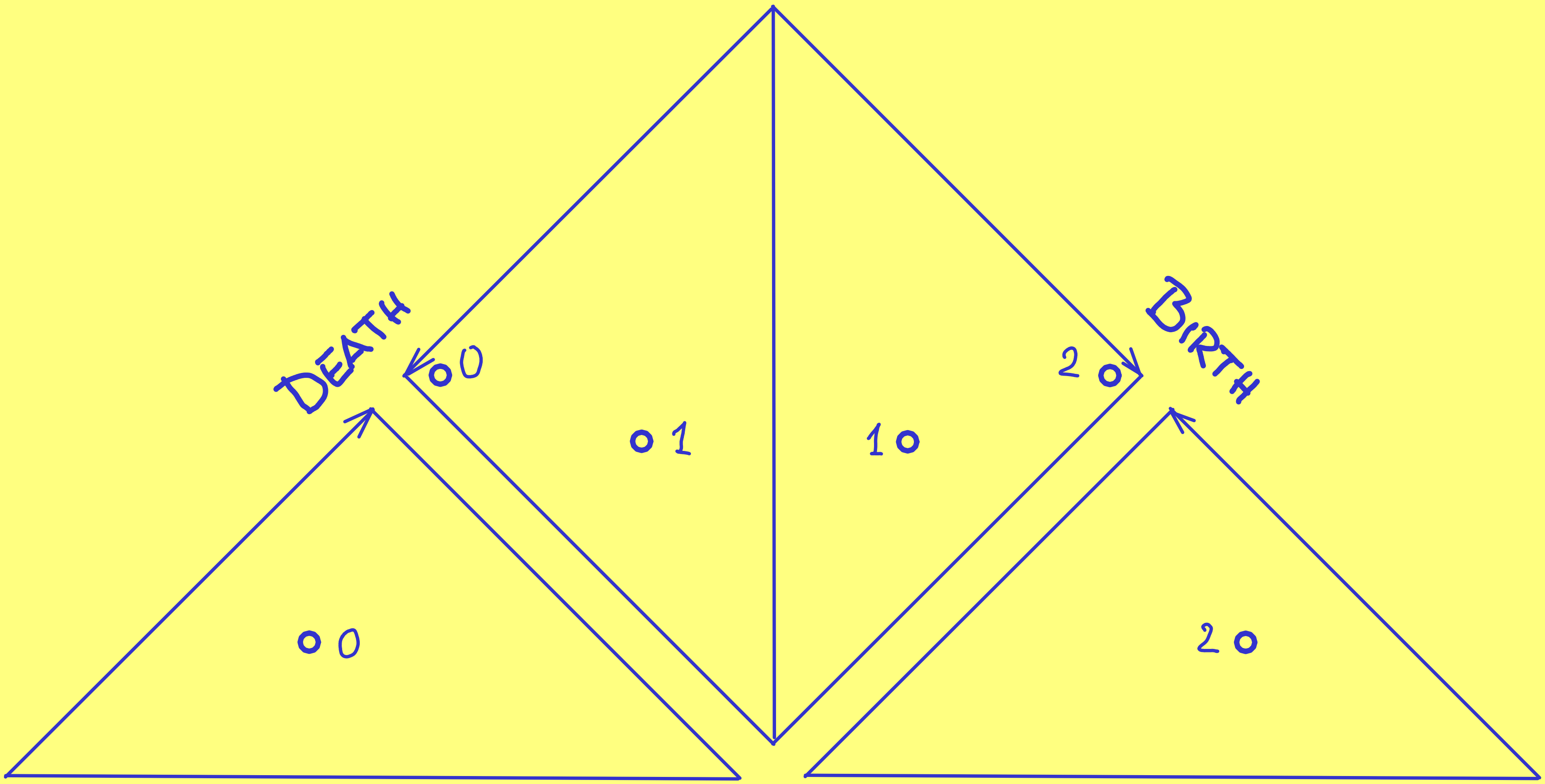

Lefschetz Duality: $\mathbb{X}$ is manifold $\Rightarrow \mathrm{Dgm}(f) = \mathrm{Dgm}^T(f)$.

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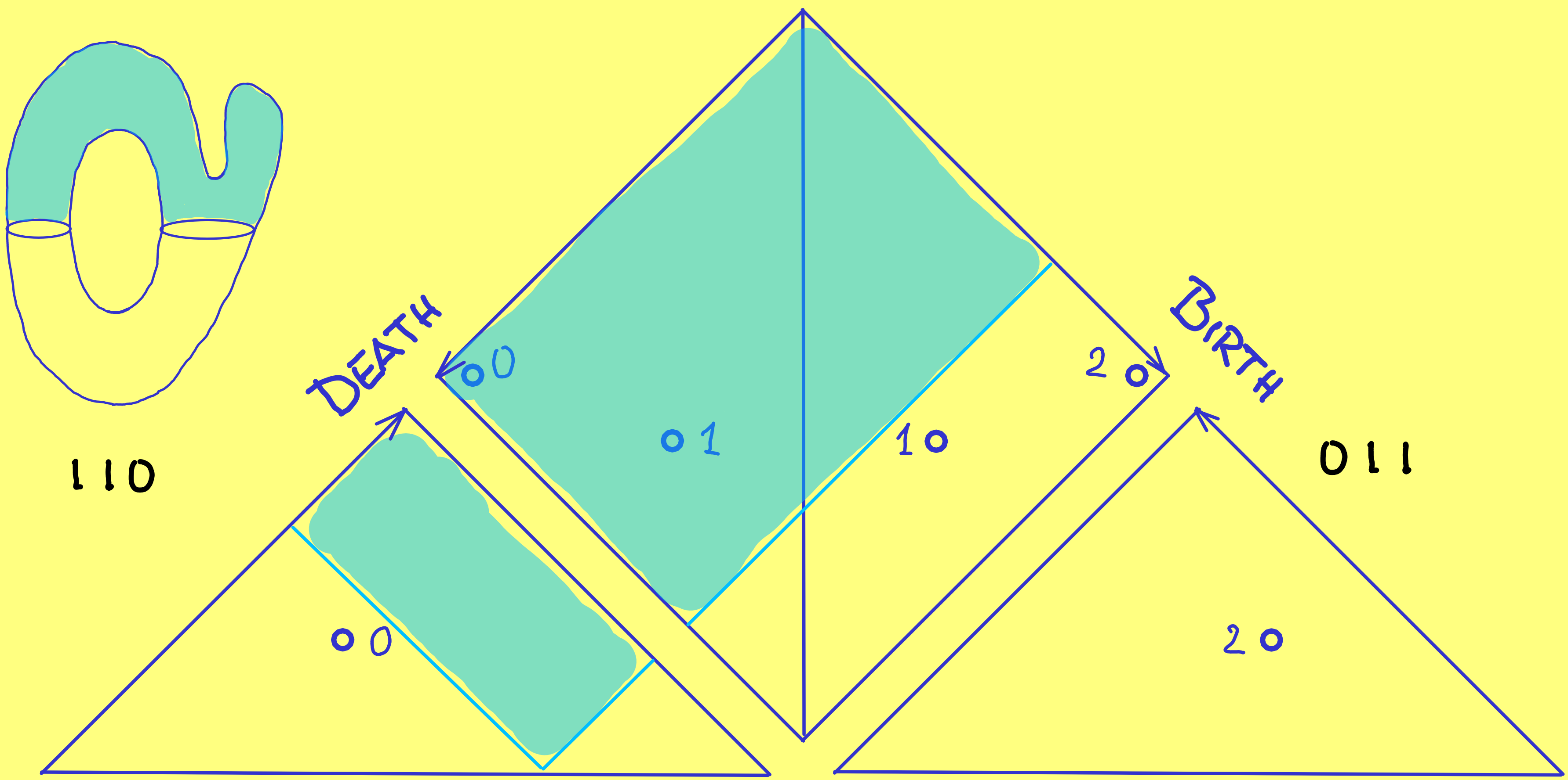

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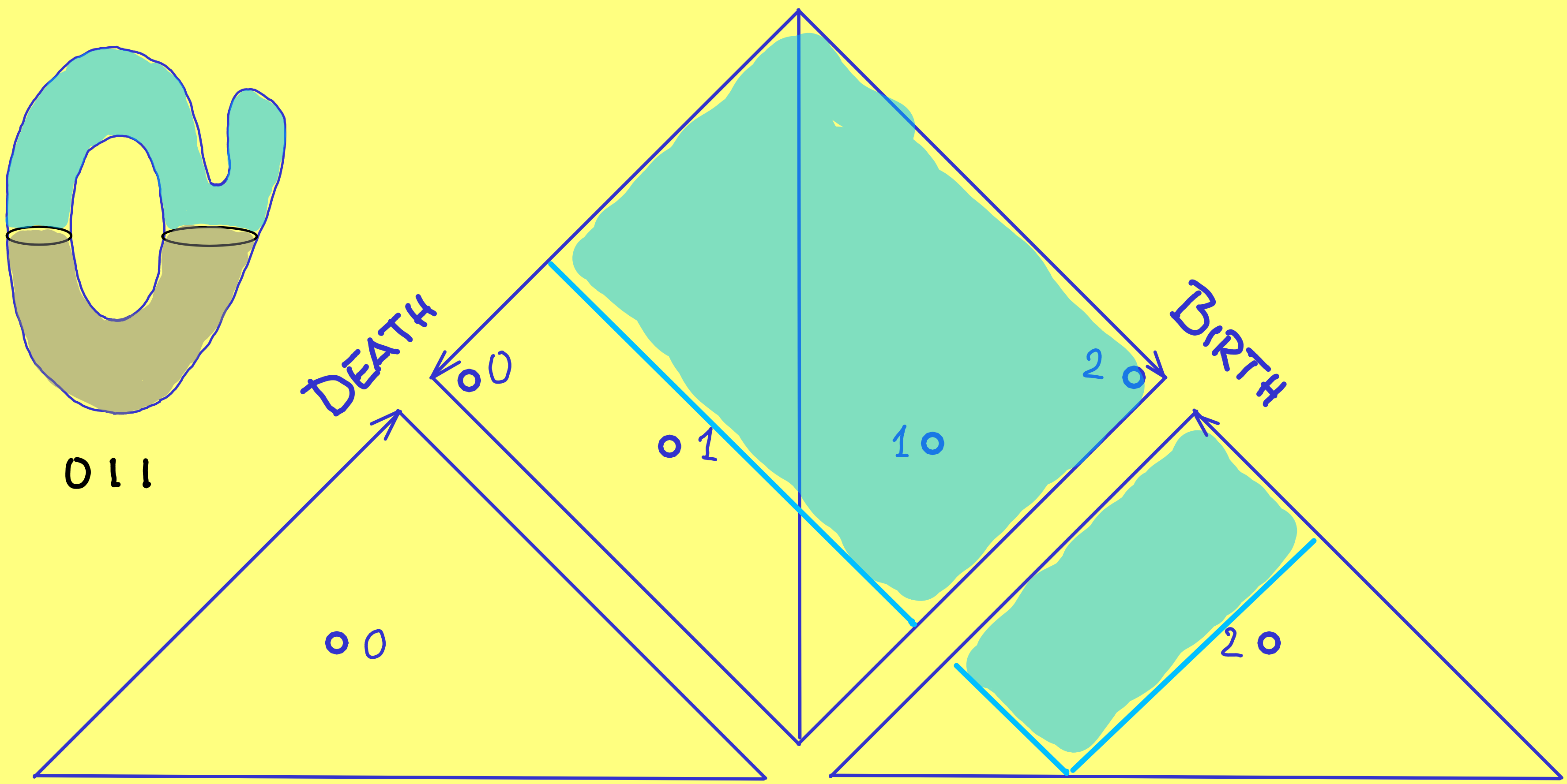
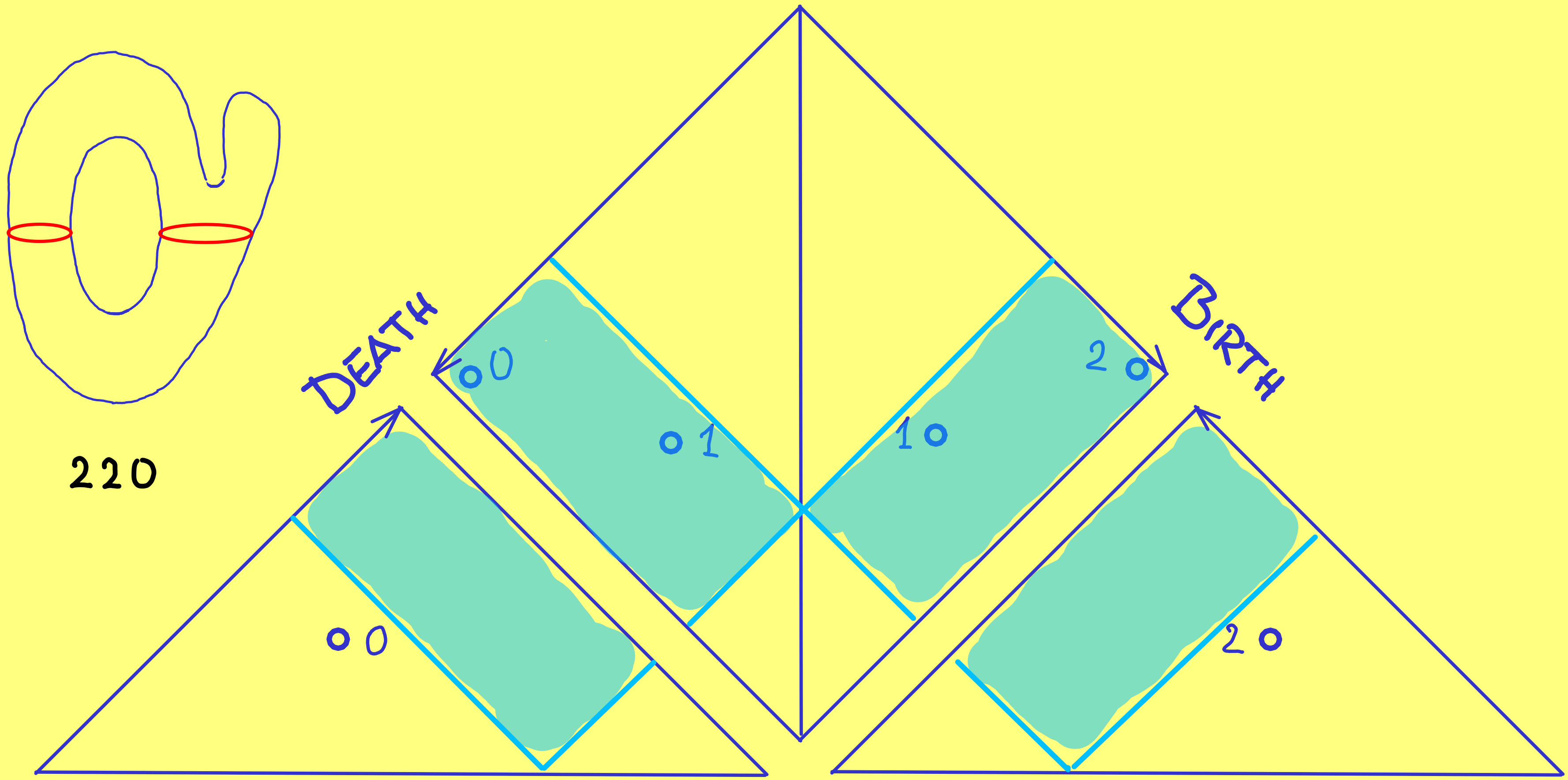


II.3 Persistence Diagram



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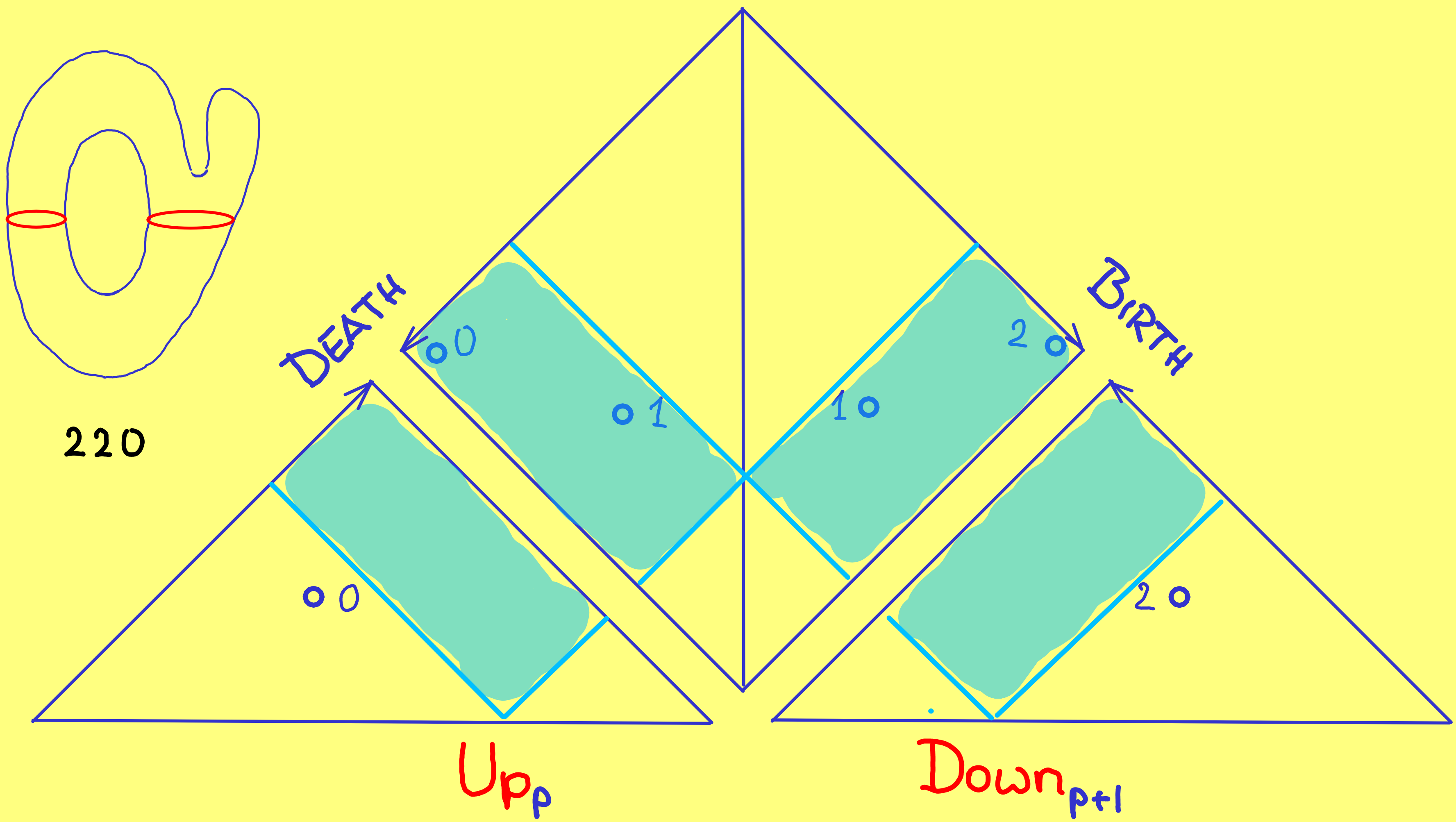
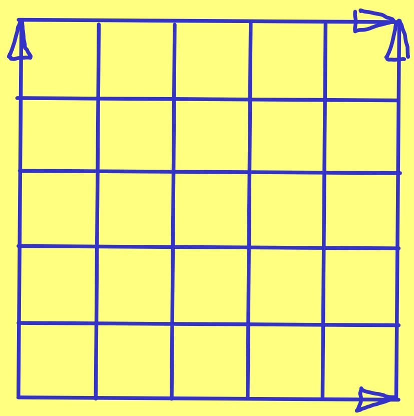
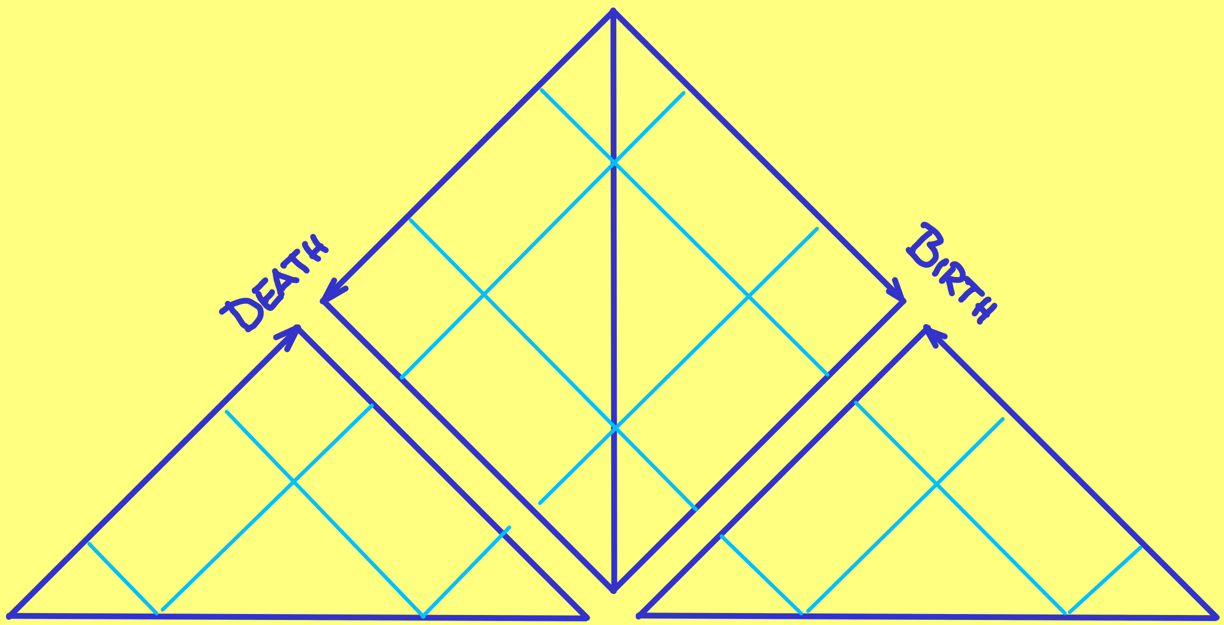


$$\beta_p(\text{level set}) = \#Up_p + \#Down_{p+1}$$

Persistence	I	Hierarchy
Extended Persistence	**II**	**Adaptive Topology**
Stability	III	Measuring
Moments	IV	Scale Space

II.4 Violation of Lefschetz Duality





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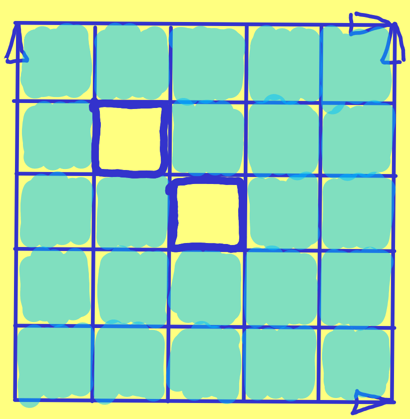

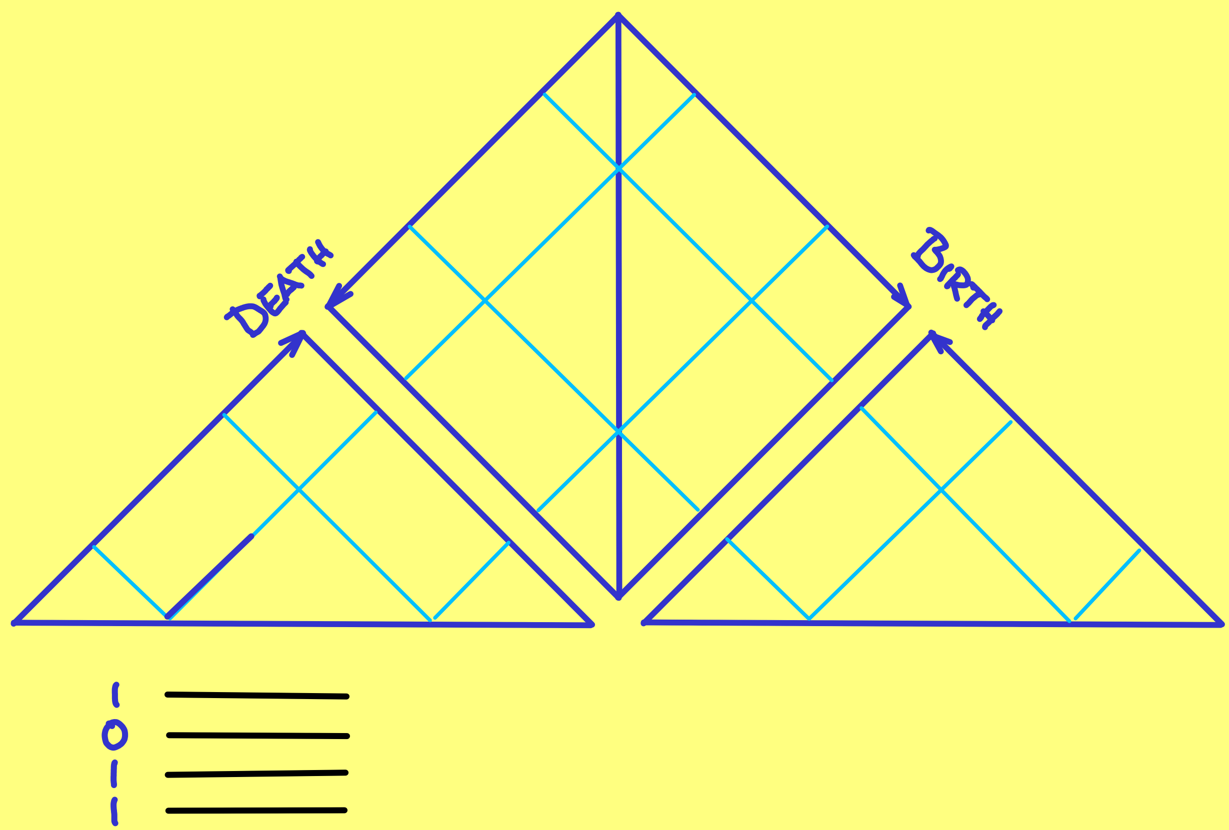

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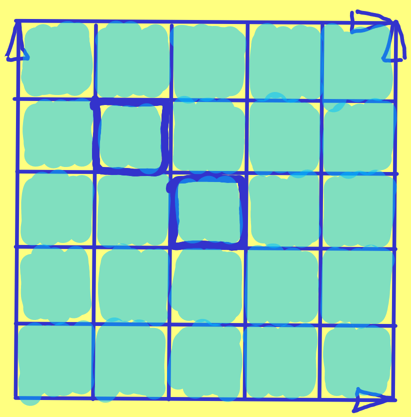

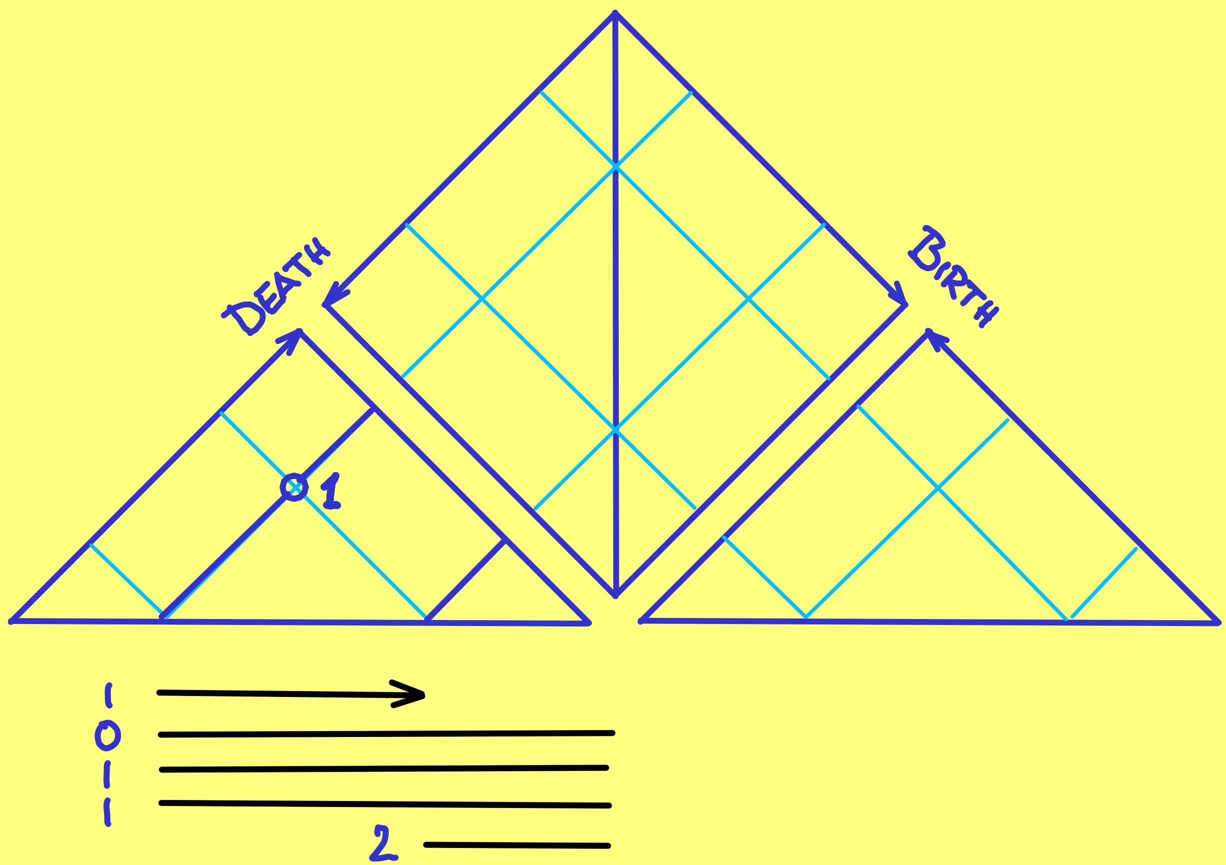

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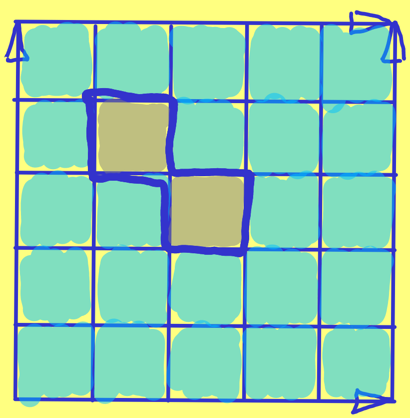

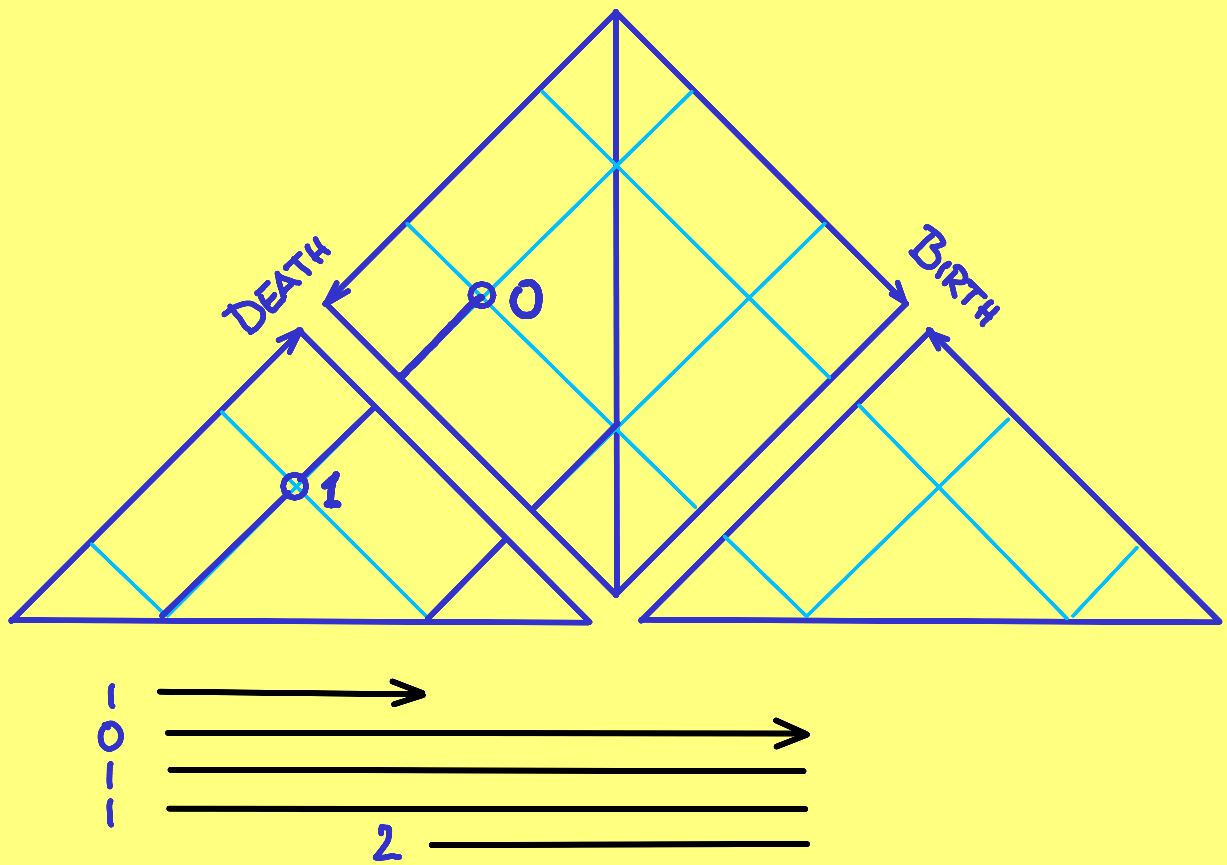

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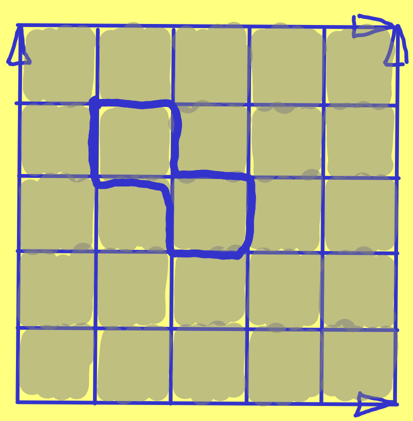
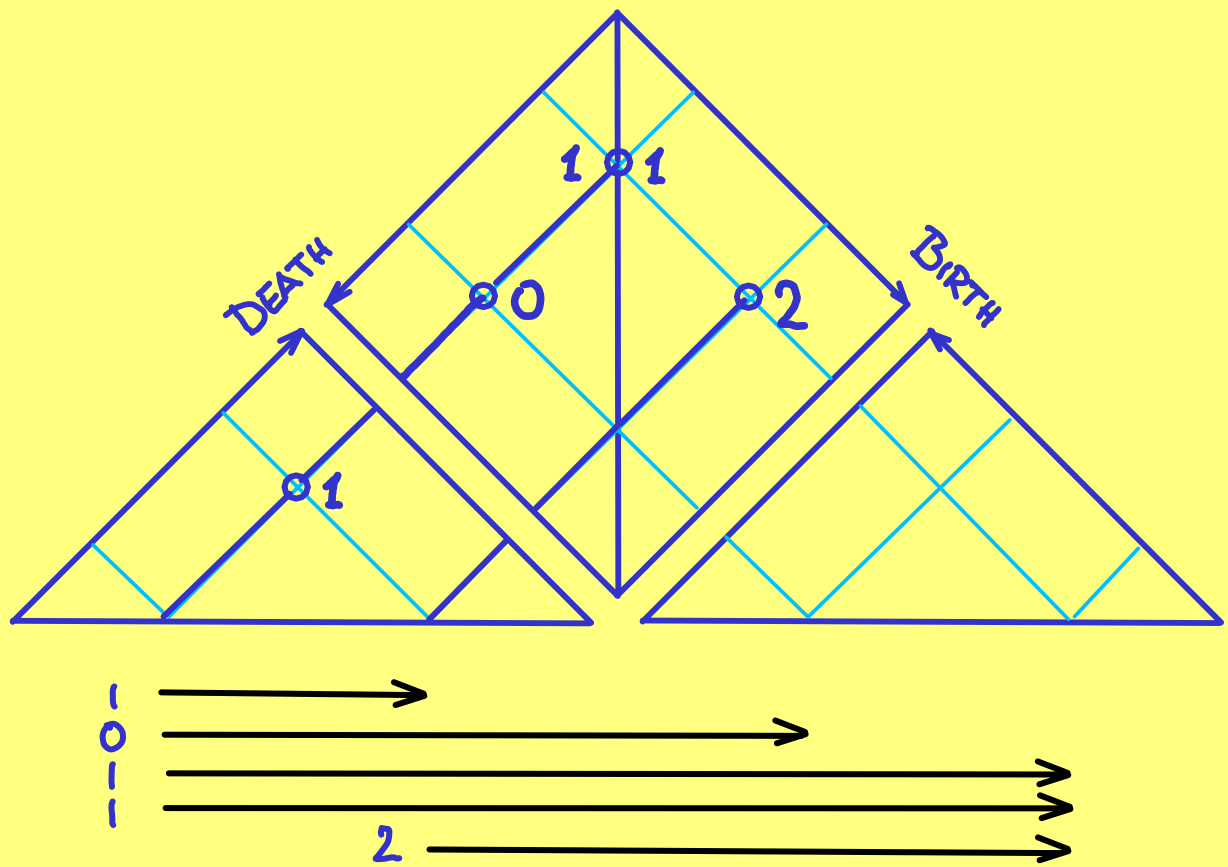
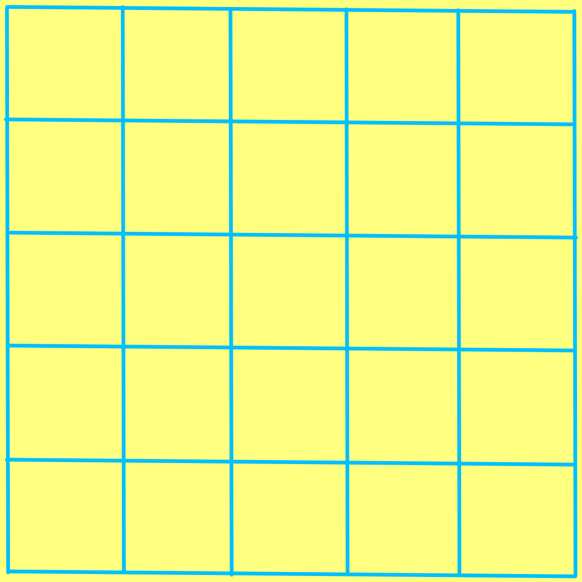




II.5 Adaptive Complex



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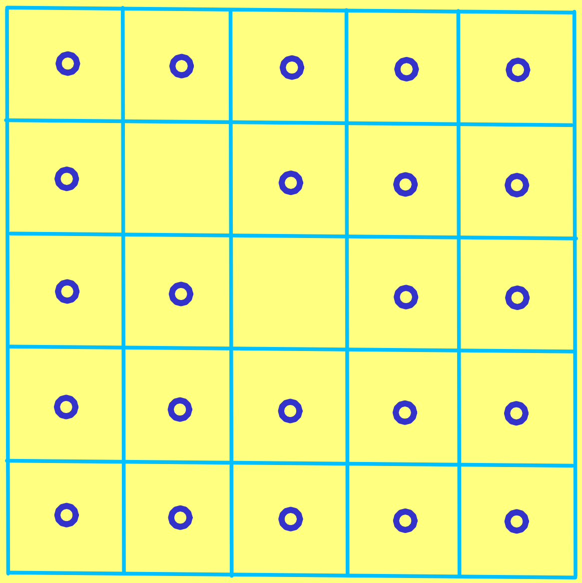

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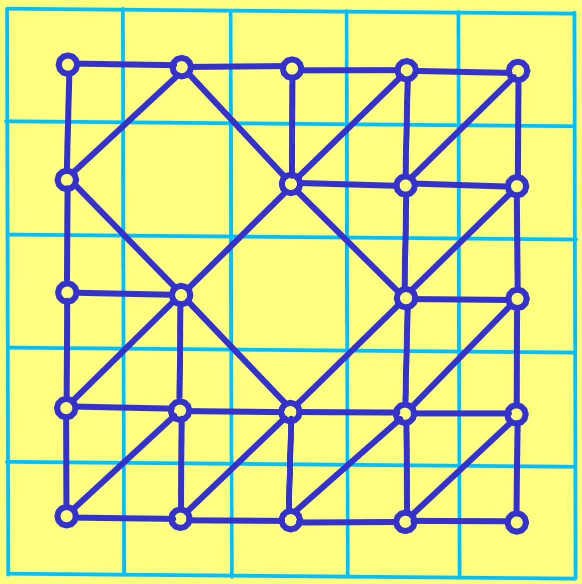

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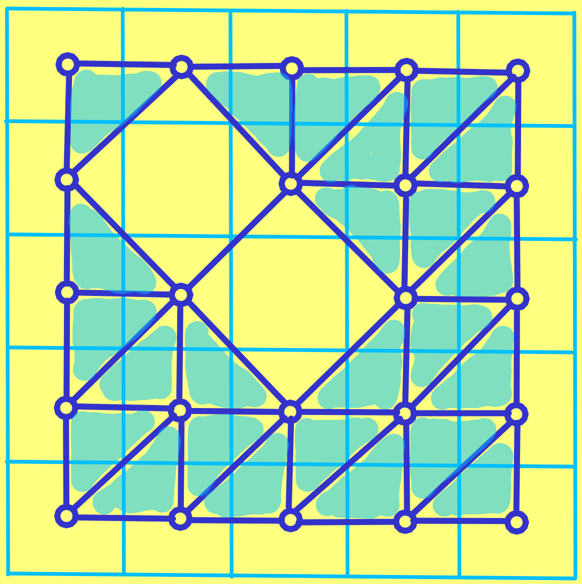

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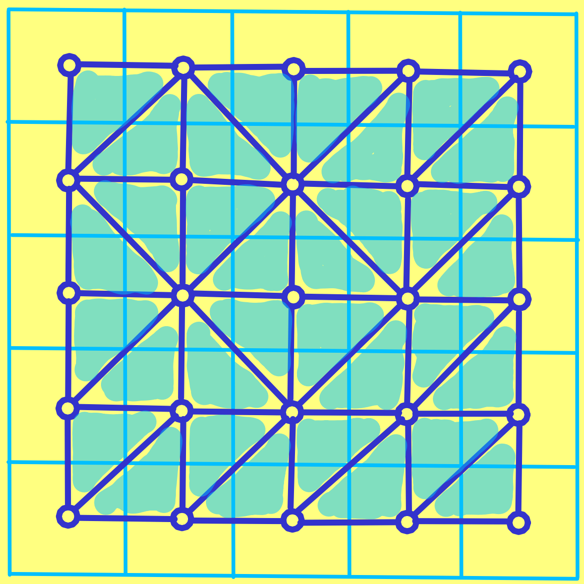

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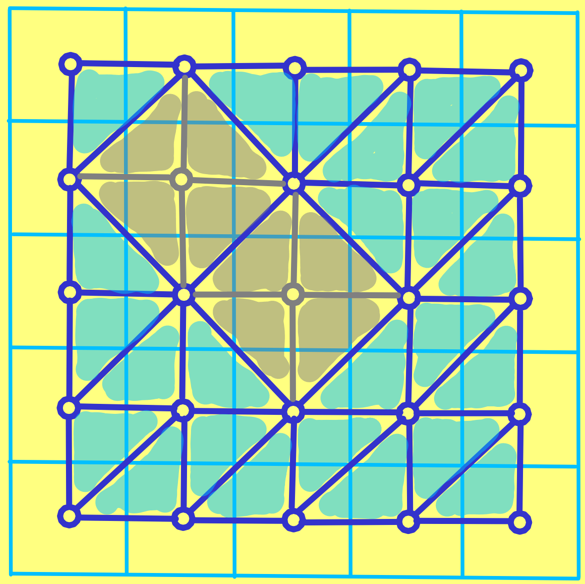

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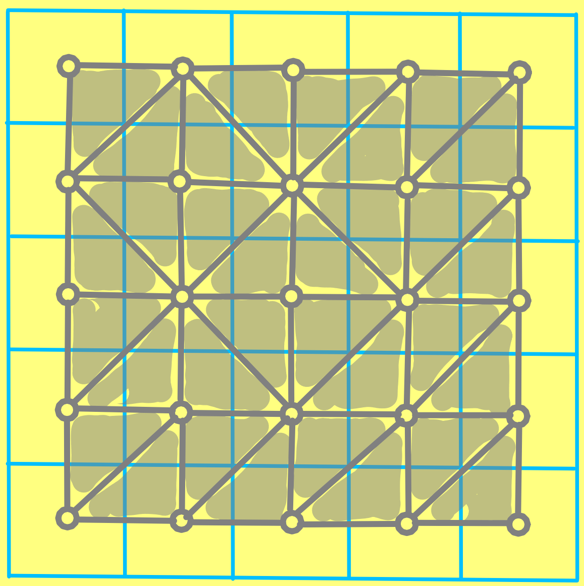

Persistence	I	Hierarchy
Extended Persistence	II	Adaptive Topology
Stability	**III**	Measuring
Moments	IV	Scale Space

III.1 Distances

III.1 DISTANCES

$f, g : \mathbb{X} \to \mathbb{R}$. The q-th Wasserstein distance between their diagrams is

$$W_q(f, g) = \inf_{\gamma : \mathrm{Dgm}(f) \to \mathrm{Dgm}(g)} \left[\sum_{x \in \mathrm{Dgm}(f)} \| x - \gamma(x) \|_\infty^q \right]^{\frac{1}{q}} .$$

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The bottleneck distance is

$$W_\infty(f,g) = \inf_{\gamma : \mathrm{Dgm}(f) \to \mathrm{Dgm}(g)} \max_{x \in \mathrm{Dgm}(f)} \| x - \gamma(x) \|_\infty$$

III.2 Bottleneck Stability

Thm. $\mathbb{X}$ triangulable, $f, g : \mathbb{X} \to \mathbb{R}$ tame.

Then $W_\infty(\mathrm{Dgm}(f), \mathrm{Dgm}(g)) \leq \| f - g \|_\infty$.

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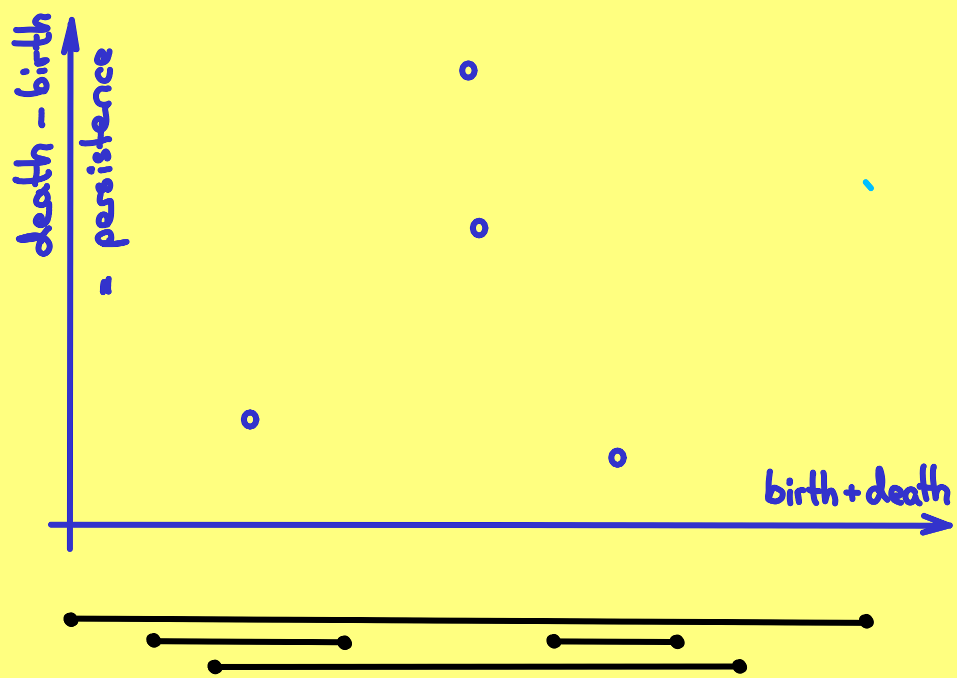

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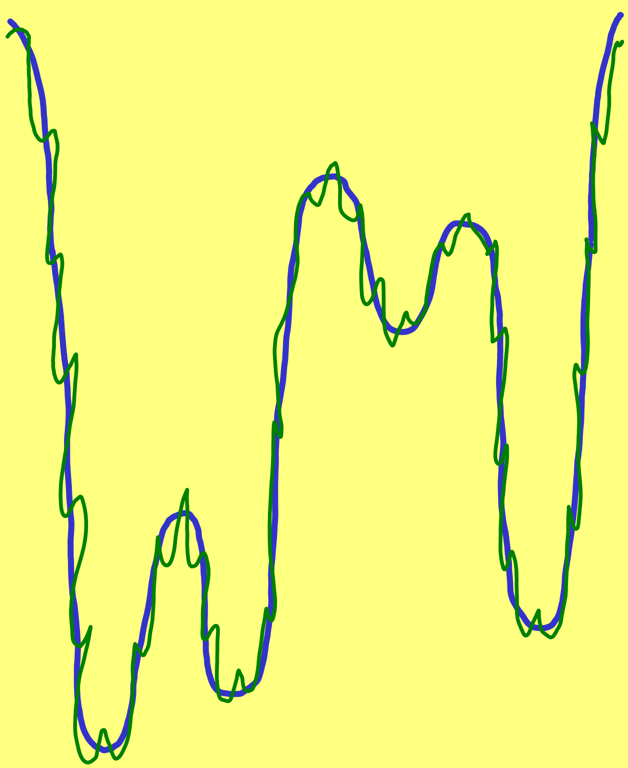

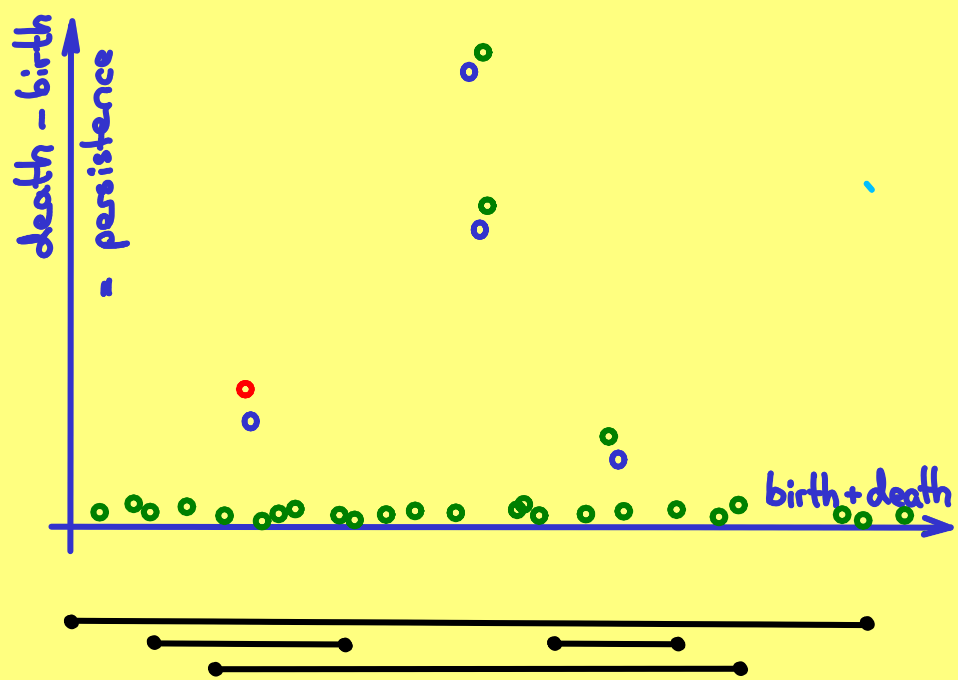

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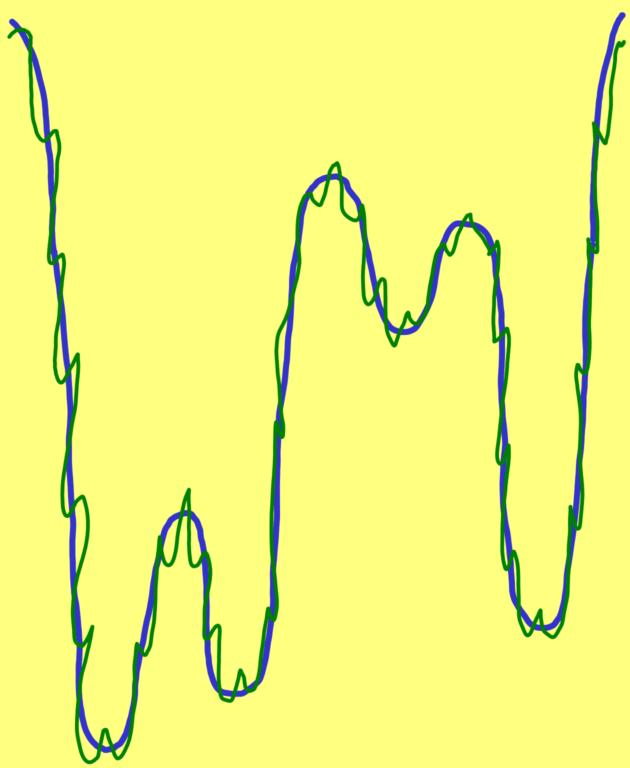
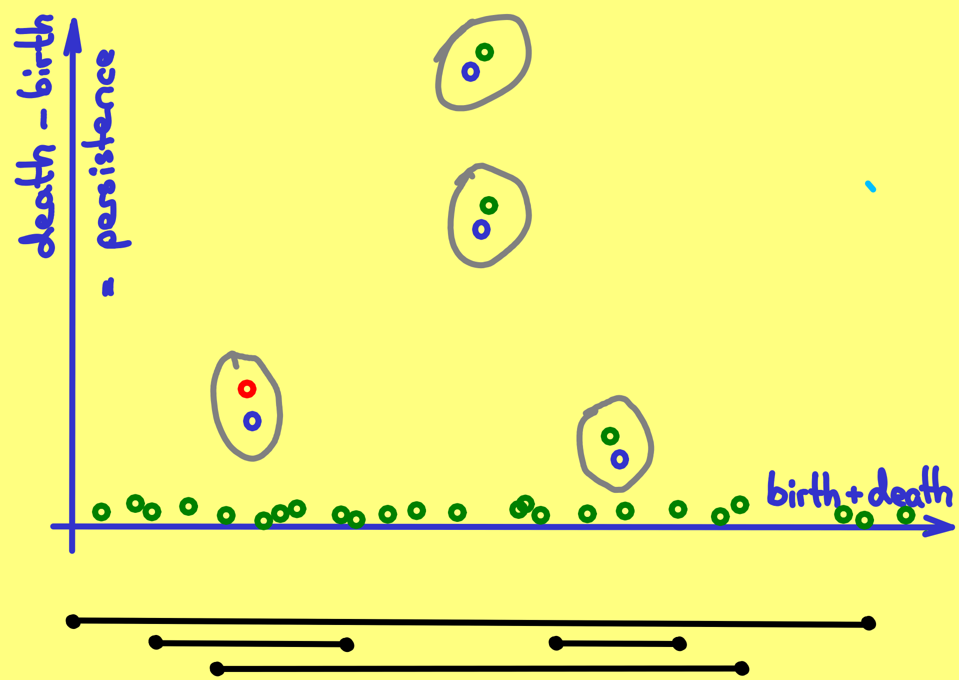




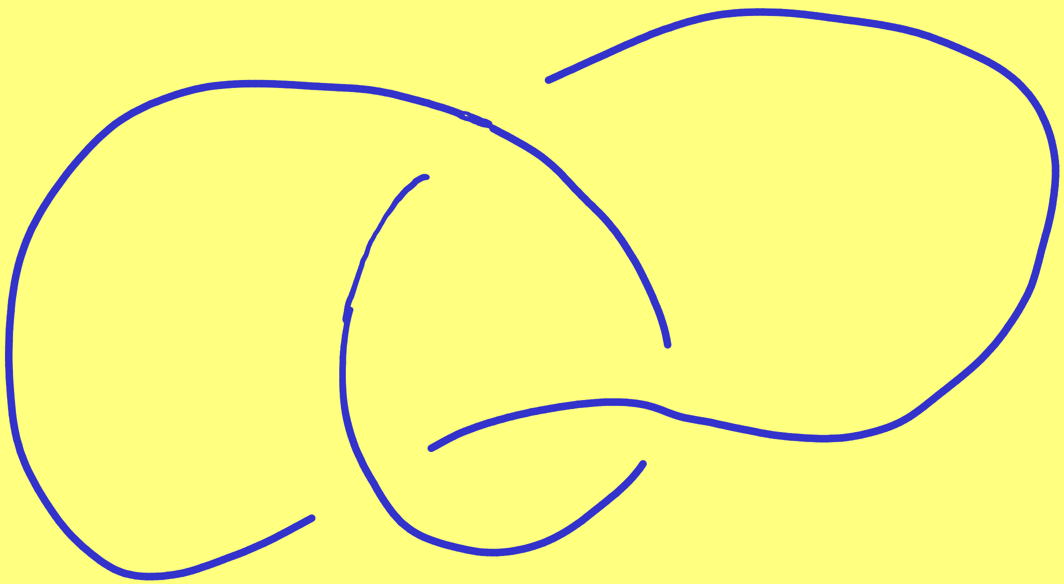
Persistence I Hierarchy

Extended Persistence II Adaptive Topology

Stability **III Measuring**

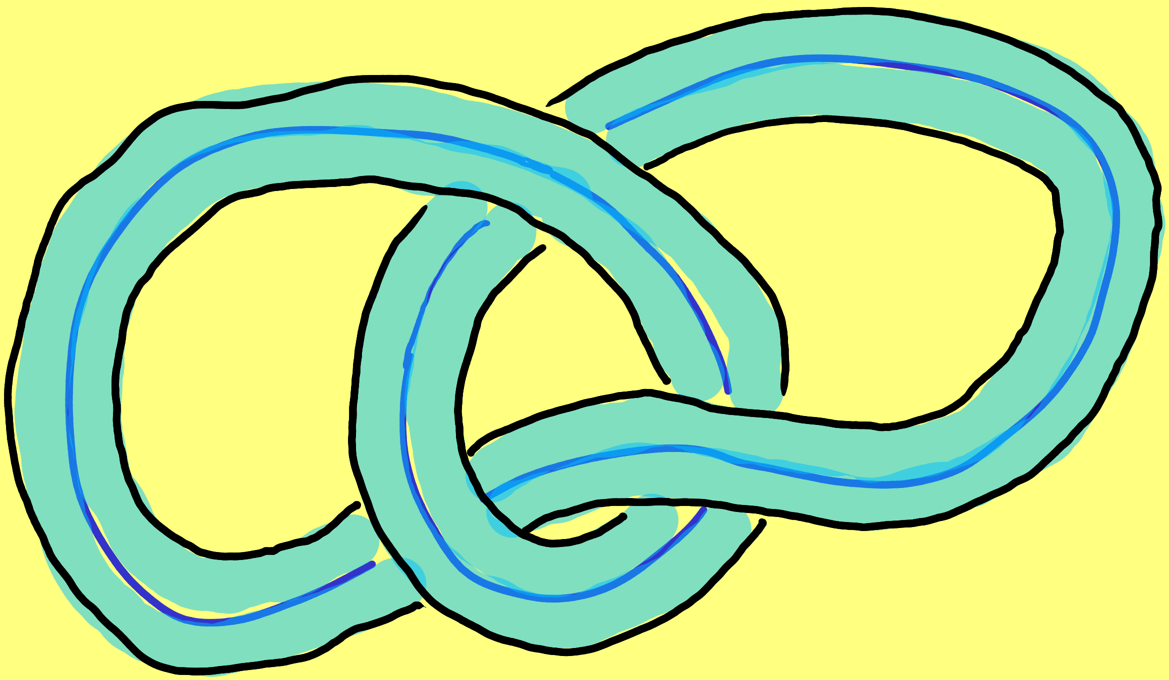
Moments IV Scale Space

III.3 TUBE



γ

III.3 TUBE



tube $\gamma_{r+\varepsilon}$

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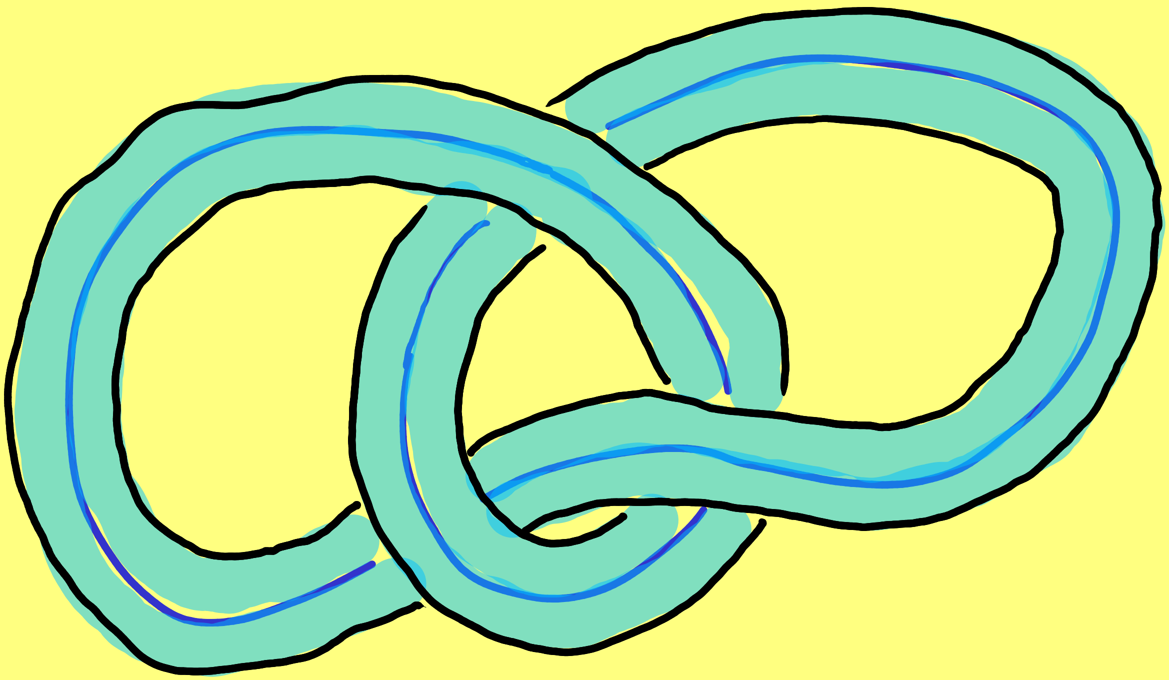
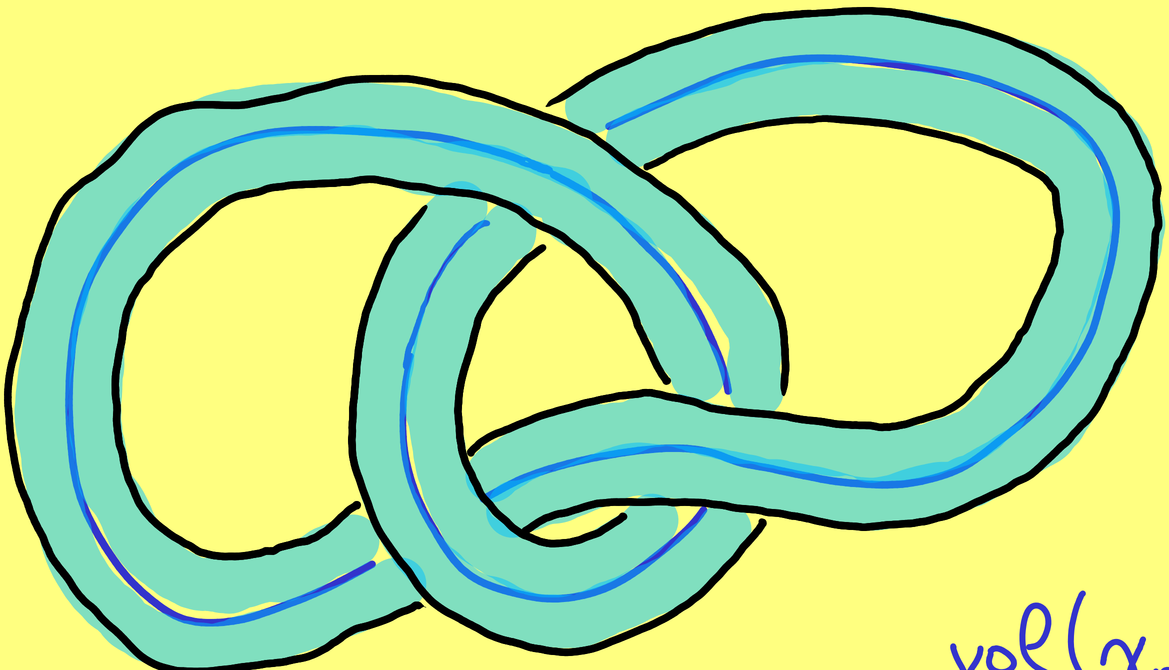


tube $\gamma_{r+\varepsilon}$

$L = \text{length}(\gamma)$

III.3 TUBE

[Weyl 1939]



tube $\gamma_{r+\varepsilon}$

$L = \text{length}(\gamma)$

$$\text{vol}(\gamma_{r+\varepsilon}) = L \cdot (r+\varepsilon)^2 \pi$$

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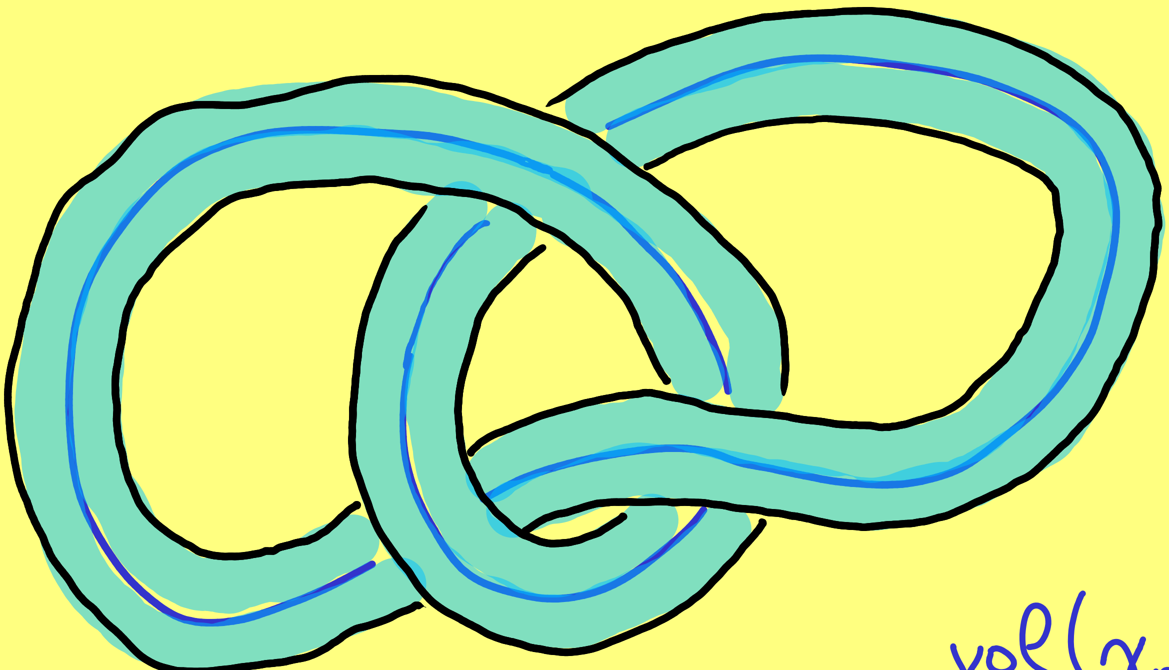


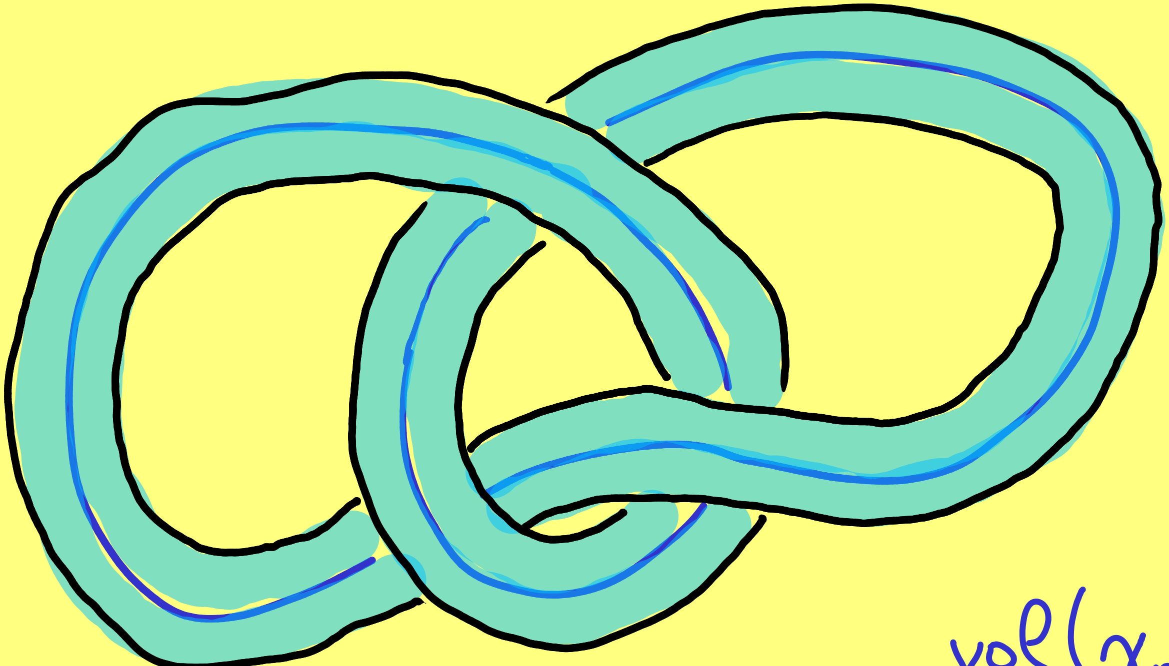
tube $\gamma_{r+\varepsilon}$

$L = \text{length}(\gamma)$

$$\mathrm{vol}(\gamma_{r+\varepsilon}) = L\cdot(r+\varepsilon)^2\pi$$
$$= Lr^2\pi + 2Lr\pi\varepsilon + L\pi\varepsilon^2$$

III.3 TUBE

[Weyl 1939]



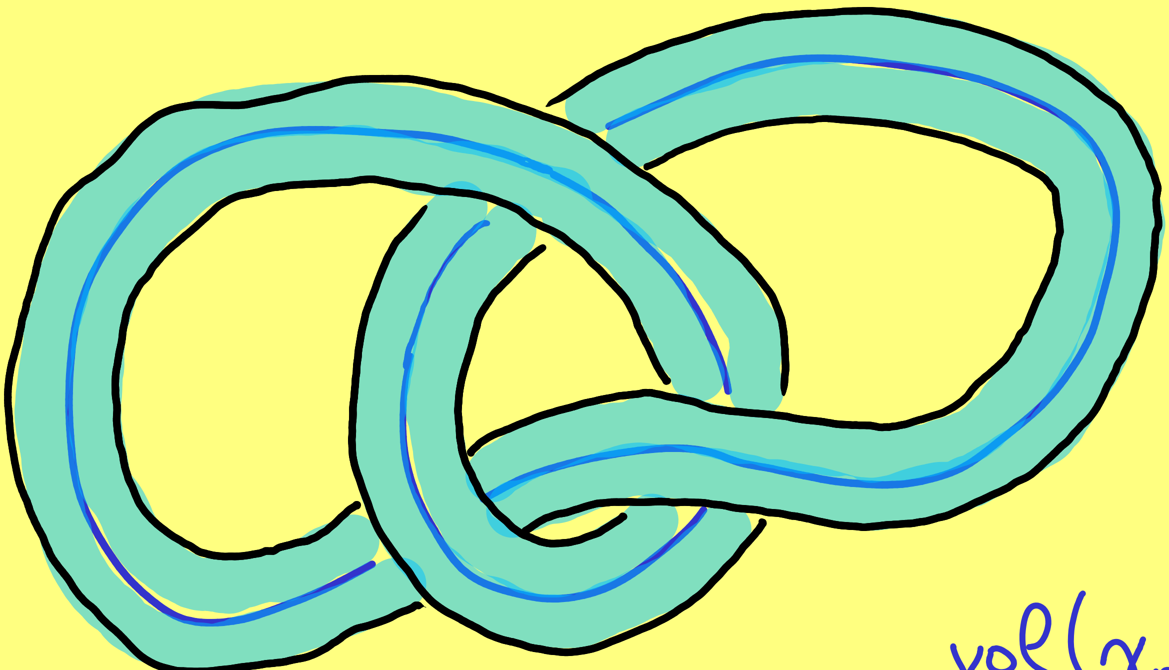
tube $\gamma_{r+\varepsilon}$

L = length(γ)

$$\begin{aligned} \mathrm{vol}(\gamma_{r+\varepsilon}) &= L\cdot(r+\varepsilon)^2\pi \\ &= Lr^2\pi + 2Lr\pi\varepsilon + L\pi\varepsilon^2 \\ &= \underset{\substack{\shortparallel \\ \text{volume}}}{Q_0} + \underset{\substack{\shortparallel \\ \text{area}}}{Q_1}\varepsilon + \underset{\substack{\shortparallel \\ \text{mean curv.}}}{Q_2}\varepsilon^2 \end{aligned}$$

III.3 Tube

[Weyl 1939]



tube $\gamma_{r+\varepsilon}$

$L = \text{length}(\gamma)$

$$\begin{aligned} \operatorname{vol}(\gamma_{r+\varepsilon}) &= L\cdot(r+\varepsilon)^2\pi \\ &= Lr^2\pi + 2Lr\pi\varepsilon + L\pi\varepsilon^2 \\ &= Q_0 + Q_1\varepsilon + Q_2\varepsilon^2 \end{aligned}$$

" volume, " area, " mean curv.

$\Longrightarrow \quad L = Q_2/\pi.$

III.4 Quermassintegral

solid shape $\mathbb{M}$ in $\mathbb{R}^3$

III.4 QUERMASSINTEGRAL

solid shape $\mathbb{M}$ in $\mathbb{R}^3$

$$\mathrm{vol}\,\mathbb{M}_\varepsilon = Q_0 + Q_1\varepsilon + Q_2\varepsilon^2 + Q_3\varepsilon^3,$$

where $$Q_i = c_i \int_{i\text{-planes } P} \chi(P \cap \mathbb{M})\, dP$$

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$$Q_2 = \int_{2\text{-planes } P} \chi(P \cap \mathbb{M})\, dP$$

$$= \frac{1}{2} \int_{u \in S^2} \qquad du$$

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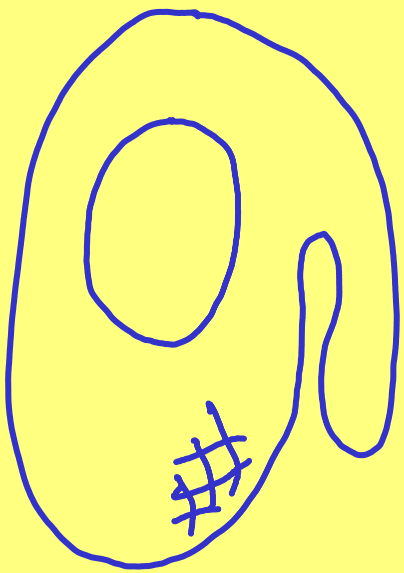

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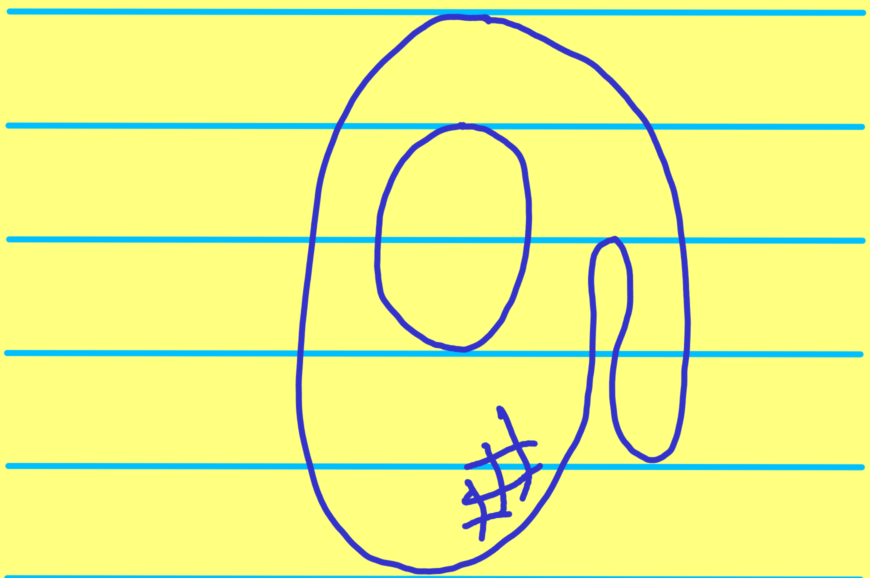

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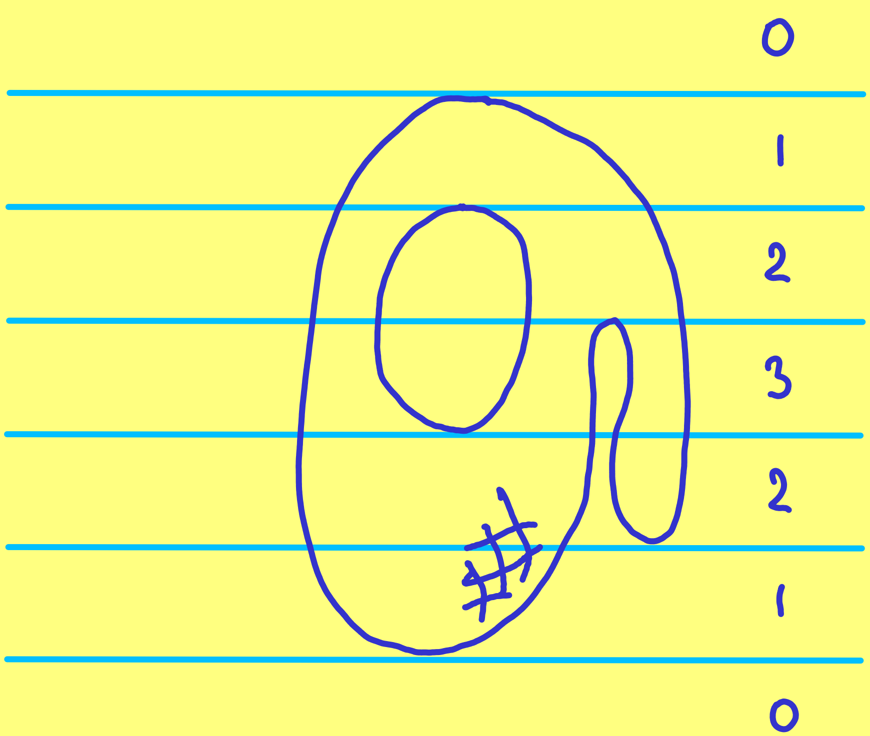

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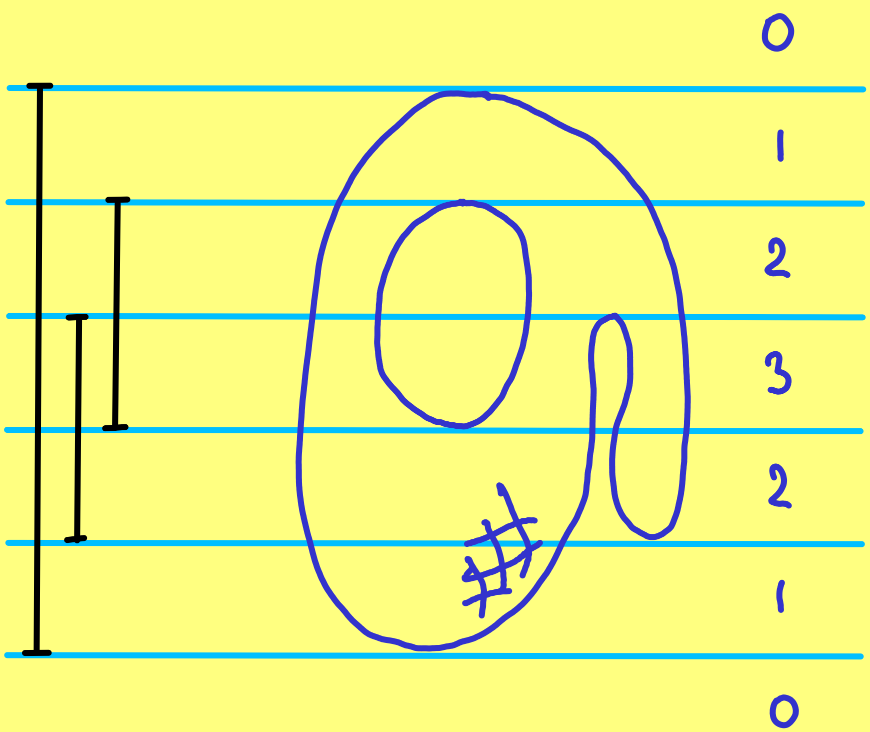

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Persistence	I	Hierarchy
Extended Persistence	II	Adaptive Topology
Stability	III	Measuring
Moments	**IV**	Scale Space

IV.1 k-TH MOMENT

The k-th p-dim. level set moment is

$$B_p^k(f) = \sum_{A \in Up_p(f)} \mathrm{pers}(A)^k + \sum_{A \in Dn_{p+1}(f)} \mathrm{pers}(A)^k .$$

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The total mean curvature is

$$Q_2(M) = \frac{1}{2} \int_{u \in \mathbb{S}^2} \sum_{p \geq 0} (-1)^p B_p^1(f_u) \, du .$$

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The total mean curvature is

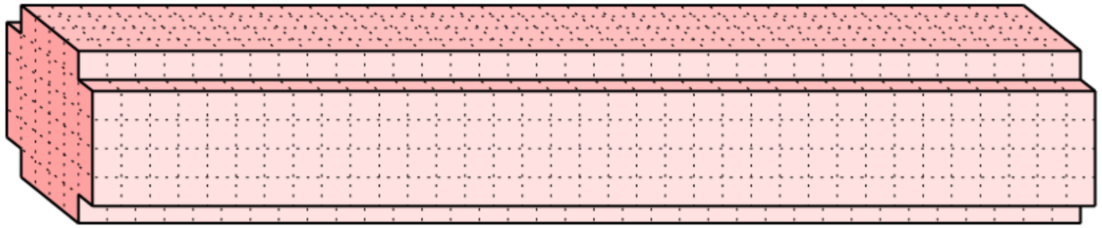
$$Q_2(M) = \frac{1}{2} \int_{u \in \mathbb{S}^2} \sum_{p \geq 0} (-1)^p B_p^1(f_u)\, du .$$

THM. f, g Lipschitz and k large enough

$$\Longrightarrow \left| B_p^k(f) - B_p^k(g) \right| \leq \mathrm{const.} \, \|f-g\|_\infty .$$

IV.2 Toy Models

DMC	qPS	qDS$_l$	qDS$_k$
47	45.38	46.96	45.78



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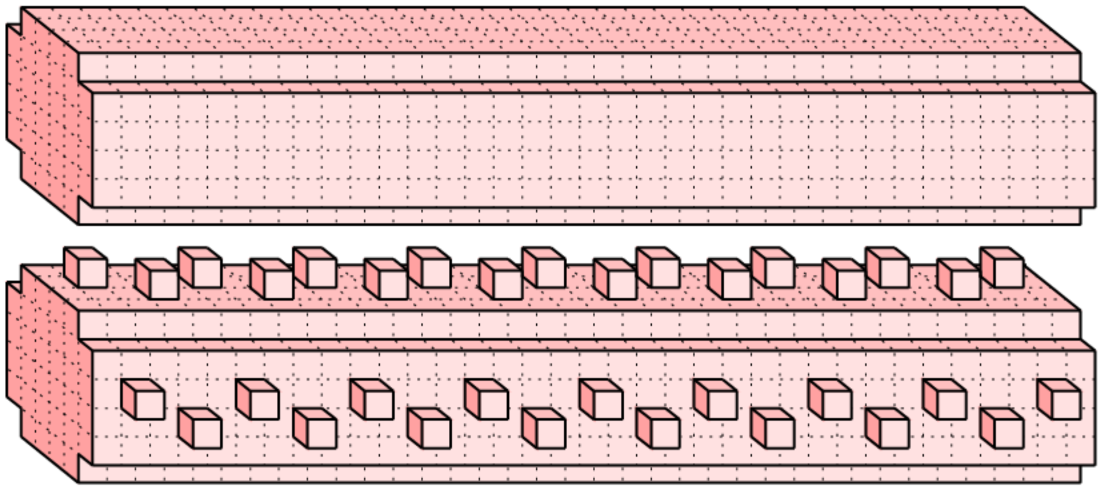
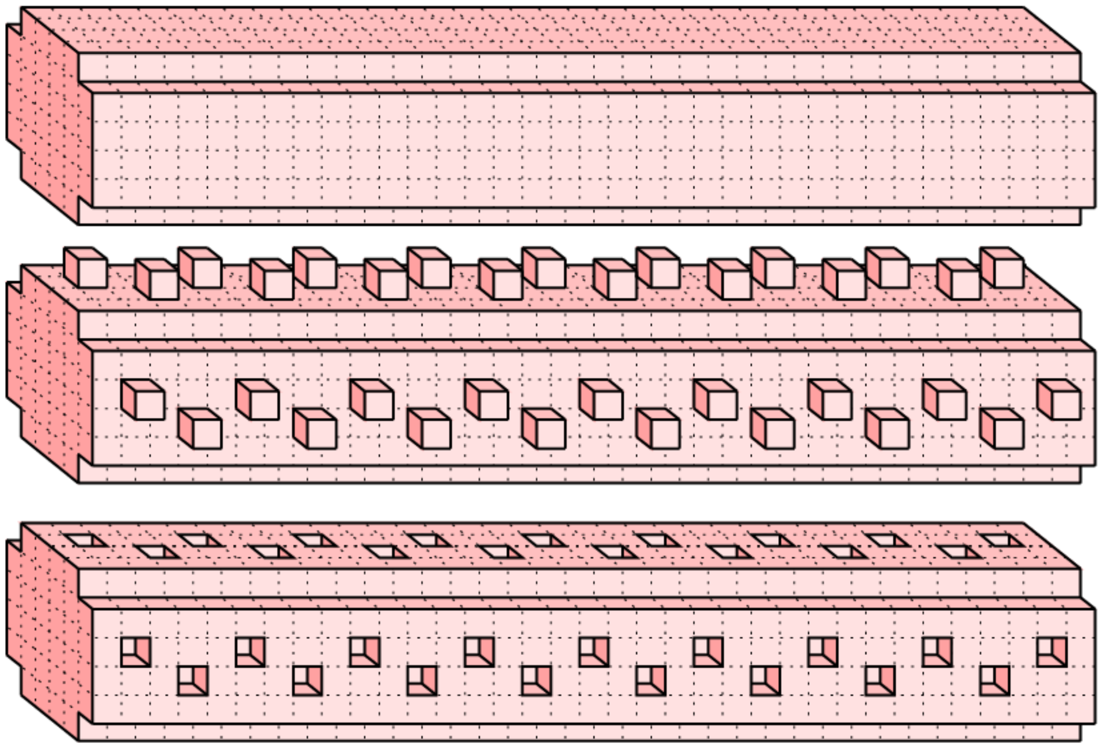


IV.2 Toy Models

DMC	qPS	qDS_l	qDS_k
47	45.38	46.96	45.78
81	82.79	81.08	46.47
13	11.49	13.26	45.36



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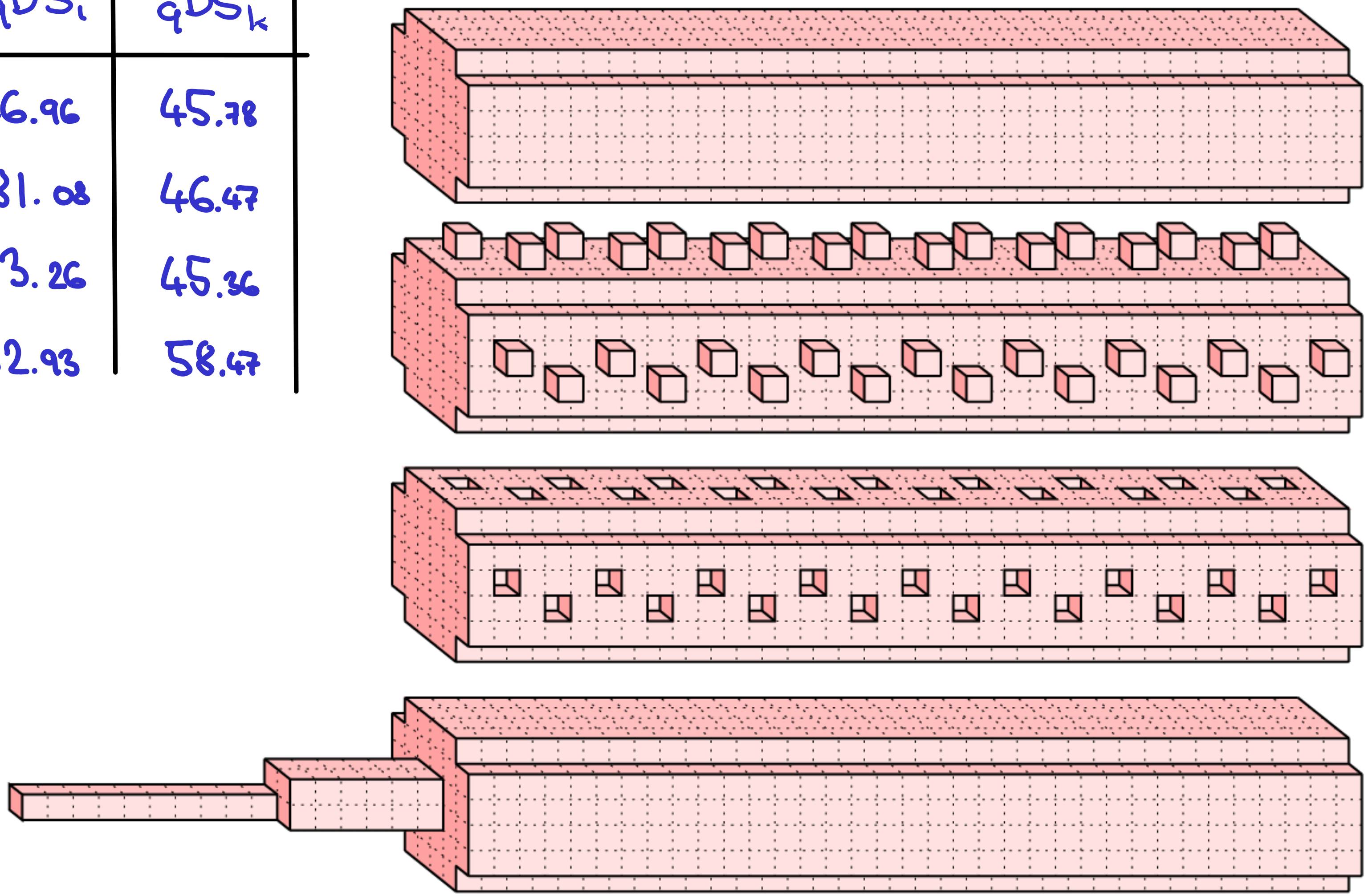
81	82.79	81.08	46.47
13	11.49	13.26	45.36
63	61.72	62.93	58.47



Persistence	I	Hierarchy
Extended Persistence	II	Adaptive Topology
Stability	III	Measuring
Moments	**IV**	**Scale Space**

IV.3 Gaussian Convolution

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Scale space of f is family f_t, for $t \geq 0$.

IV.4 Loss of Persistence

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THM. $f : \Omega \to \mathbb{R}$ for compact set $\Omega \subseteq \mathbb{R}^2$

IV.4 LOSS OF PERSISTENCE

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s.t. f_t is tame for all $t \geq 0$.

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Thm. $f : \Omega \to \mathbb{R}$ for compact set $\Omega \subseteq \mathbb{R}^2$ s.t. f_t is tame for all $t \geq 0$. Then the q-th moment of $\mathrm{Dgm}(f_t)$ is at most $\frac{\text{const}}{t}$

IV.4 Loss of Persistence

THM. $f : \Omega \to \mathbb{R}$ for compact set $\Omega \subseteq \mathbb{R}^2$ s.t. f_t is tame for all $t \geq 0$. Then the q-th moment of $\mathrm{Dgm}(f_t)$ is at most $\frac{\text{const}}{t}$ for all $q > \frac{5+\sqrt{17}}{2}$.

Collaborators

Collaborators

David Letscher

Afra Zomorodian

Collaborators

David Letscher

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THANK YOU